Big Bang Cosmology Fundamentals

{**Abstract:** *This is part one of a two part series on the Big Bang theory. Here we cover the foundational concepts used in the standard model we cover in part 2. We start out with the basic Cosmological Principle for an isotropic and homogeneous Universe. We then review Hubble's Law that came from the discovery that the Universe was expanding. We go into some depth to illustrate what expanding space is and how it impacts the basic idea of 'distance'. This includes a definition of Cosmic Distance and how it leads to the Visible Horizon. We then develop a concept of how the Universe's expansion would work using Newton's gravitational theory including his Shell Theorem. We use this to define a cosmic scale factor and use it to see what happens in a matter dominated Universe. We then expand that to include a radiation dominated Universe. With the Newtonian mechanics view in hand, we update to Friedmann's equation based on Einstein's General Theory of Relativity along with the Equation of State. We examine the impact of flat, spherical and hyperbolic space on the cosmic sale factor, and identify the Critical Energy Density needed in order to have flat space. We end with a look at cosmological redshift, and an observation on galaxy counts that lead to the conclusion that we exist in flat space-time.* **}**

Introduction

Hello and welcome to the How Old Is It chapter on the Universe. Here we will cover the current best fit theory for how the Universe has evolved to its current state. It's called the Lambda Cold Dark Matter model with a period of rapid inflation in its earlies times.

It's the standard model of the Big Bang cosmology because it explains the Universe we see today in great detail including:

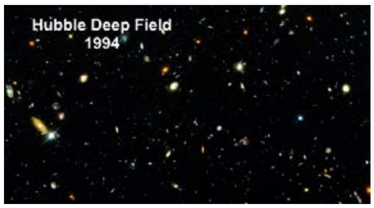
- the web-like large scale structure of the current Universe
- the existence and structure of the cosmic microwave background radiation
- the accelerating expansion of the Universe
- the amount of hydrogen we see throughout the cosmos

We'll go over the meaning of all these terms, and as always with these videos, we'll cover the available evidence each step of the way. I trust you'll find it fascinating and informative. We'll start with the Cosmological Principle.

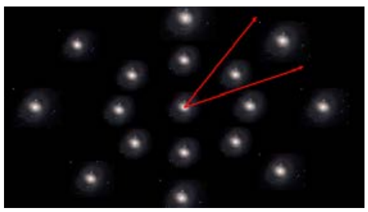
Cosmological Principle

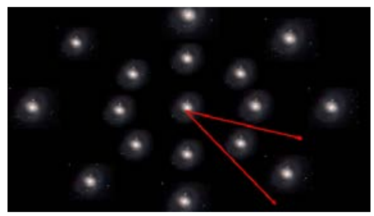
[Music: Puccini - La Boheme - Che gelida manina - what a cold little hand]

When we look out into the Universe far enough, we see, on average, the same distribution of galaxies in all directions.

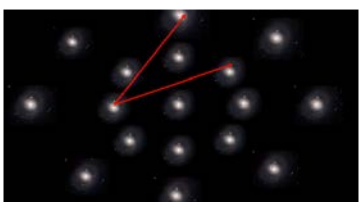


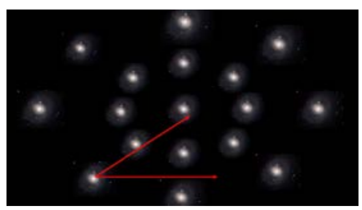
In other words, there is no preferred direction in space. We call it isotropic.





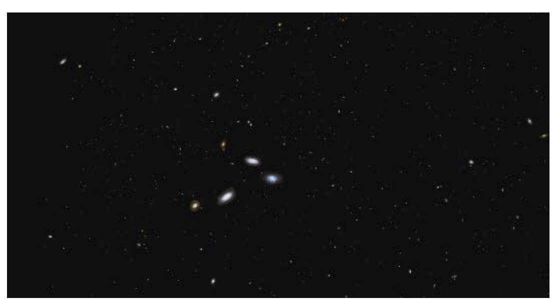
But being isotropic for us doesn't necessarily mean it is isotropic for viewers in far off locations. It is possible to imagine galaxy configurations where things look the same in all directions from one point of view, and not from another. But this is unlikely.





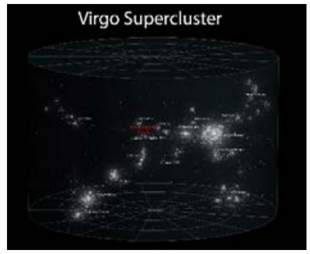
Our assumption is that all observers will see the same isotropic distribution of galaxies. In other words, there is no preferred location in the Universe. We call it homogeneous. Galaxies densities are the same everywhere (as long as we are using large enough distances and volumes).

This is the Cosmological Principle. It simply states that the Universe is both isotropic with no preferred direction and homogeneous with no preferred place. All directions are equal. And all places are equal as far as the laws of physics are concerned.

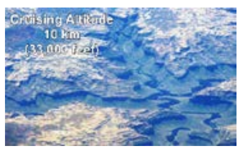


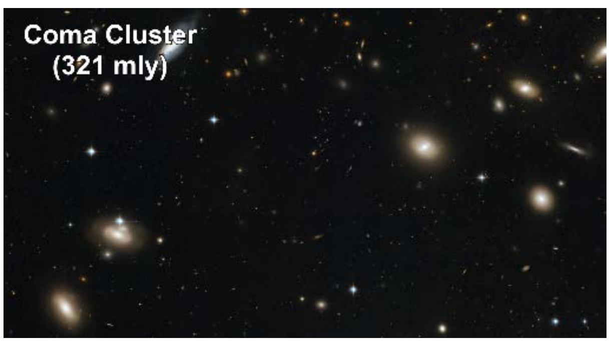
[By the way, it is interesting to note that these symmetries in space lead to key conservation laws. Conservation of linear momentum is derived from space symmetry, and conservation of angular momentum is derived from directional symmetry.]

In the "How Far Away Is It" video book, we saw that galaxies clump up into galaxy clusters and that these galaxy clusters clump up into superclusters. So to get to this cosmological principle, we need to be talking about distances of many billions of light years.



For comparison, it's like saying the surface of the Earth is a sphere. But up close, we see mountains and valleys galore. It isn't clear that we have a sphere until we get far enough away for the nearby structures to average out. We have to go out 45,000 km to actually see the sphere.

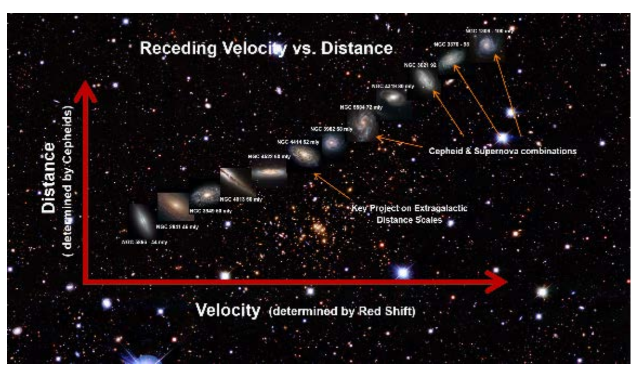




For the cosmos, we find that there are some very large structures like The Great Sloan Wall of galaxy clusters (including the great Coma Cluster). The wall is around 1.8 bly long. So the Cosmological Principle is actually an approximation and we can expect deviations from predicted galactic behaviors due to perturbations created by these kinds of extreme structures that do not fit a homogeneous model.

Hubble' Law

For thousands of years, it was taken for granted that the Universe was static. It always was as we see it now, and it always will be. This was the case when Newton developed his gravitational equations, and it was the case when Einstein developed his General Theory of Relativity. Then in 1929, Edwin Hubble published his studies of galaxy velocities. He found that, except for a few near-by galaxies, all the spectra shifts were to the red. All of them were moving away from us. Here's what we see from galaxies in our Virgo Supercluster out to a hundred million light years.



He had discovered that the Universe was expanding away from us. His Hubble Law says that the further away a galaxy is, the faster it is receding away from us. The relationship is linear (a straight line), so the equation is simple: The receding velocity of a galaxy is equal to its distance times a constant now called the Hubble Constant.

v = HR

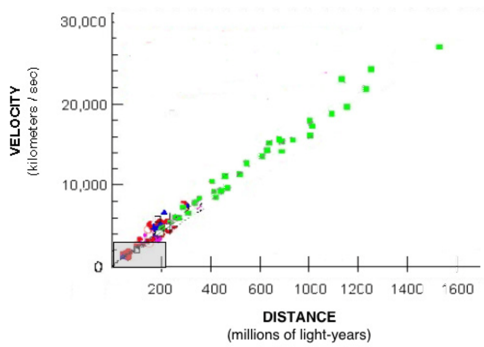
Where

R = distance

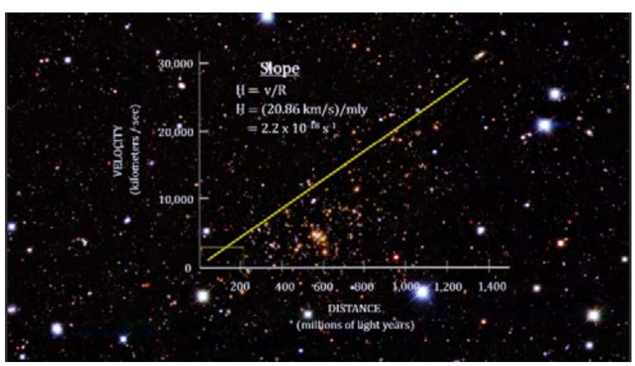
v = velocity

H = Hubble Constant

This constant has been refined over time, and the distances used have increased by orders of magnitude using tools like the Hubble Space Telescope to analyze Type 1a Supernovas out to billions of light years. The box at the lower left shows the region that Edwin Hubble probed.

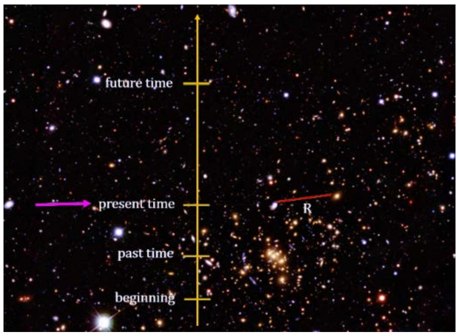


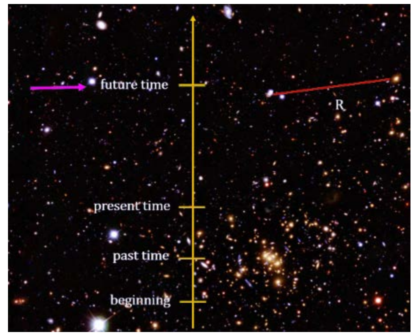
The current best value for the Hubble Constant using this approach is 20.86 km/s/mly (that's around 13 miles/sec per million-light-years) [68 km/s/Mpc]. That is, the receding velocity of a galaxy goes up by almost 21 km/sec for each additional million light years away from us it is from us. [In basic units, that's $2.2\text{x}10^{-18}\text{s}^{-1}$.] This slow and steady movement of galaxies away from us is called the Hubble Flow.



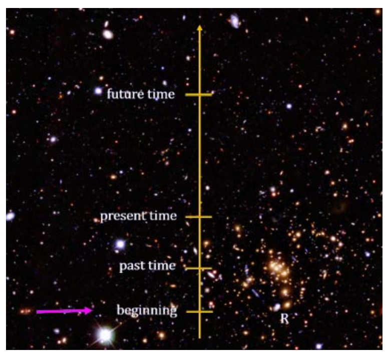
The Expanding Universe **[Music: Beethoven - Piano Concerto No 2]**

Given this flow of galaxies away from us, consider our galaxy, galaxy A, separated by a large distance from galaxy B. In galaxy A's frame of reference, it is at rest. Galaxy B is moving away. Its distance from the Milky Way will continue to increase as time goes on.

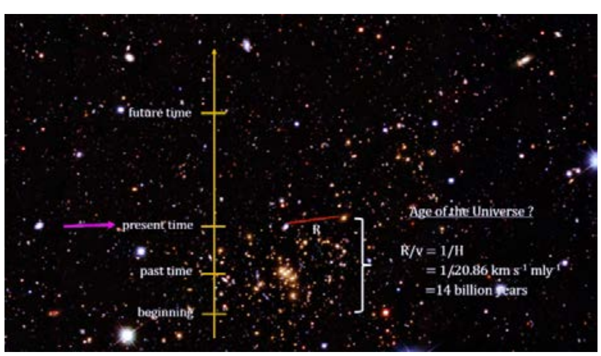




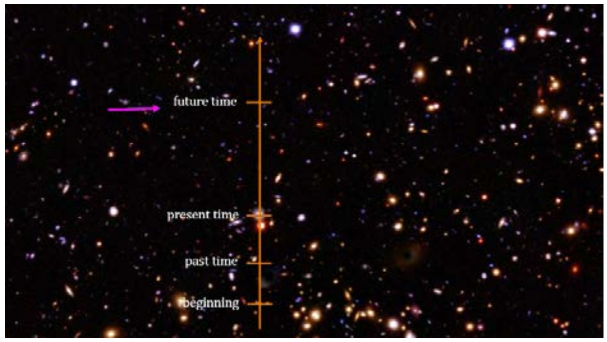
It follows that going backwards in time, galaxy B was getting closer to the Milky Way. We see that at some point in the distant past, they would have been extremely close to each other.



Assuming for now that the velocity is constant, we can divide it into the current distance between the two galaxies to see how long it took them to get this far apart. That's just one over the Hubble Constant. So, without actually knowing the distance between them, or their separation velocity, we find that the two galaxies would have taken 14 billion years to reach their current separation.



When we combine the Cosmological Principle with our view that all distant galaxies are moving away from us, we conclude that all galaxies are moving away from each other and that the further away a galaxy is from any other galaxy, the faster it is moving away.

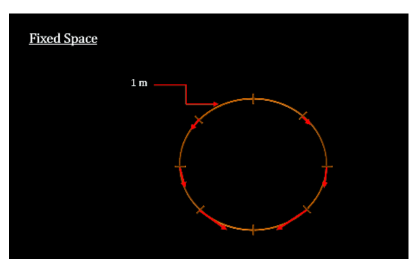


It follows that, going back in time, all galaxies in the Universe were once extremely close together, if not actually in the exact same place at the exact same time 14 billion years ago. We call the theory that tries to explain how this might have happened, the Big Bang Theory.

Expanding Space

The flow of all galaxies away from each other, with faster velocities the further away from each other they are, cannot happen in a fixed volume because, in a fixed volume, some reference frames would have to have distant objects heading towards them for others to have them moving away. It can only be explained if the space that these galaxies exist in is itself expanding. This is the base assumption of the Big Bang Theory. Here's a one dimensional example to illustrate why this is the case.

Consider an 8 meter circle with marks one meter apart. If we are the top mark, and all the other marks are moving away from us, then from other points of view, marks are getting closer. The system is not homogenous.



But if the apparent motion is due to the amount of space expanding, we get a different picture. Here the marks hold their position on the line, but the line grows. Let's say each meter on the line expands to 2 meters over the course of a minute. We see that the distance between adjacent marks goes up one meter and their apparent velocity as seen by each other is 1 meter per minute. But more distant marks have increased their distance and velocity by more than that. And the further away any two marks are, the more their distance and velocity have increased. And most importantly, this will be the same no matter which mark is used for the reference frame.

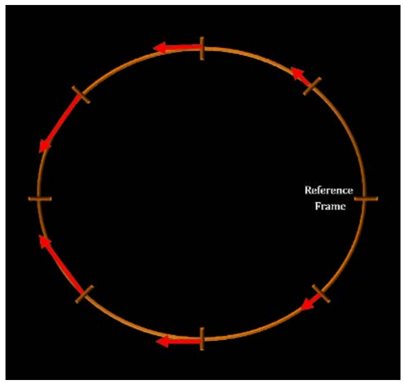
Expanding Space

If expansion = 1 m/min , then

$$\begin{aligned} K &= V/R \\ &= (1m/min)/1m \\ &= 1/m = 1/60s \\ &= 1.67 \times 10^{-2} s^{-1} \end{aligned}$$

Where

- V = velocity of a mark
- R = distance to the mark
- K = constant of proportionality



Velocity Formula

$$V = K\,R$$

- V = mark velocity
- R = mark distance
- $K = 1.67 \times 10^{-2} s^{-1}$

In order to illustrate the point, this example used an expansion rate that is 74 thousand trillion times greater than the actual expansion rate as determined by the Hubble constant.

$$\frac{K = 1.67 \times 10^{-2} s^{-1}}{H = 2.27 \times 10^{-18} s^{-1}} = 0.74 \times 10^{20}$$

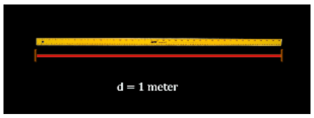
The real expansion is very slow. If we take a look at what the expansion does to one meter, we see that it would take a million years to expands by just 7 millionths of a meter. That's way too slow to ever notice or even measure in a lab in a lifetime. And it is why it's so easy to overcome it with local gravity out to the Andromeda galaxy.

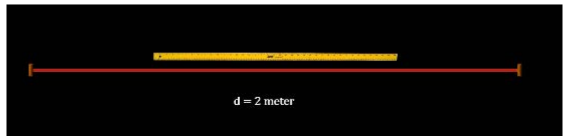
$$\begin{aligned} V &= HR \\ V &= 2.2 \times 10^{-18}\,m/s \\ \delta d &= Vt = 6.94 \times 10^{-6}\,m \end{aligned}$$

Where

- V = expansion velocity
- R = 1 meter
- $H = 2.2 \times 10^{-18} s^{-1}$
- δd = meter expansion
- t = 1 million years $= 3.145 \times 10^{12}$ s

It should be noted that this expansion of space itself does not pull apart objects that exist in that space. A meter stick does not expand. It will measure two meters where there once was only one. That's because the size of the meter stick is determined by the forces that hold it together, and these forces are not changing.

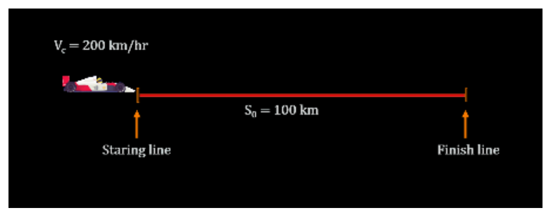




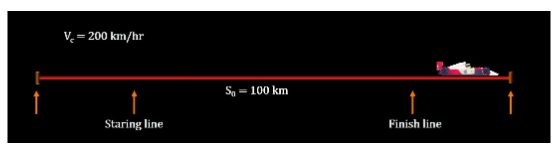
Cosmic Distance **[Music: Borodin - Polovetsian Dances From Prince Igor]**

Expanding space has significant implications for measuring distance.

Consider a racetrack that's 100 km long at the time the race car crosses the starting line traveling 200 km/hr. If we set its odometer to zero when it leaves the starting line, it will read 100 km when it reaches the finish line in half an hour.



Now suppose the track is expanding at 50 km/hr. How long will it take to reach the finish line and how far will it have traveled? It will be more than a half hour and farther than the 100 km the track started with. But it will be less than the length of the track when the car crosses the finish-line because some of the expansion will have happened to space the car had already traveled through. A little algebra gives us the exact numbers.



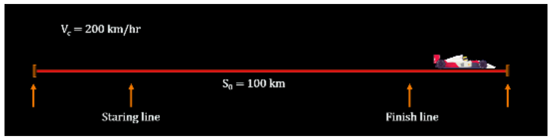
Let

S_0 =	starting distance	= 100 km
V_c =	velocity of car	= 200 km/hr
t =	time	
S =	distance car traveled	= 114.286 km
V_x =	space expansion rate	= 50 km/hr
S_x =	space expansion	
S_e =	ending track distance	

Then

$$S = S_0 + S_x/2$$
$$V_c t = S_0 + V_x t$$
$$t = S_0 / (V_c - V_x/2)$$
$$= 100\text{ km}/(200 - 25)\text{km/hr} = 0.571\text{ hr}$$
$$S_x = V_x t = 50\text{ km/hr} \times 0.5714\text{ hr} = 28.57\text{ km}$$
$$S = V_c t = 200\text{ km/hr} \times 0.5714\text{ hr} = 114.286\text{ km}$$
$$S_e = S_0 + S_x = 100\text{ km} + 28.57\text{ km} = 128.57\text{ km}$$

Now suppose we didn't know the track's distance at the start. But we did know how fast the car travels, and the car's odometer tells us how far it traveled. And furthermore, suppose we found a way to figure out that space was expanding at 50 km/hr. With that, we can figure the original track size, and we can figure out how far apart they are once the car reached the finish line.



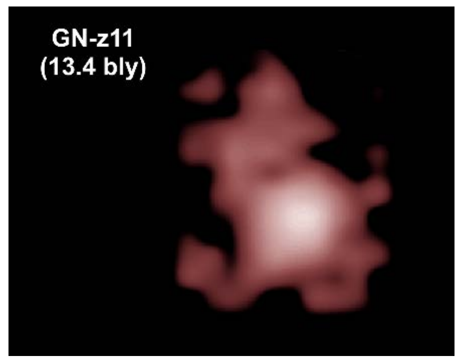
Let

S_0 = starting distance = ?
V_c = velocity of car = 200 km/hr
t = time
S = distance car traveled = 114.286 km
V_x = space expansion rate = 50 km/hr
S_x = space expansion
S_e = ending track distance

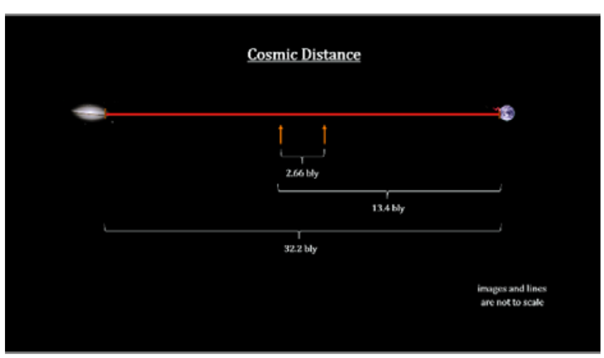
Then

$S_0 = S - S_x/2$
$t = S/V_c$
$= 114.286$ km/hr / 200 km/hr = 0.571 hr
$S_x = V_x t$
$= 50$ km/hr x 0.5714 hr = 28.57 km
$S_0 = S_e - S_x$
$= 128.57$ km – 28.57 km = 100 km

The principles are the same for light traveling to us from distant galaxies. Here we are zooming into GN-z11, the most distant object ever found. The galaxy's redshift combined with Hubble's law gives us the distance the light traveled – 13.4 billion light-years. And we know the speed of light so the time traveled was 13.4 billion years. We normally say that the galaxy is therefore 13.4 billion light years away.



But, during its long travel time, space expanded considerably. In fact, GN-z11 was less than 2.7 billion light years away from us when the light started its journey. And the galaxy is now over 30 billion light years away. In order to calculate these distances, we need to know how the Universe expanded during the light's journey. It's not like the simple constant we used in the car racetrack example. In fact, we'll be spending the remainder of this video on just how and why our Universe's expansion behaves the way it does. And we'll return to GB-z11 along the way.

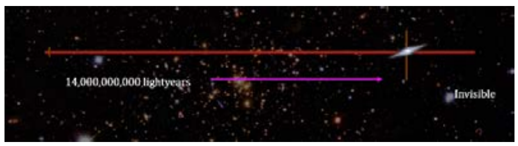


Visible Horizon

Note that, if a galaxy is far enough away, its apparent velocity will be faster than the speed of light – and its light would never reach us. It would be beyond the physical visible horizon for the Universe. It's not that it is moving through space that fast, it's just that more space is being created per second between us and them than light can travers in one second. Plugging in the numbers, we find that all galaxies beyond 14 billion light years could never be seen here. GN-z11 is now 32 billion lightyears away, so the light that is leaving GN-z11 now, can never reach us.

Let

V_c = speed of light = 3×10^8 m/s
$H = 2.27 \times 10^{-18} s^{-1}$
R_h = visible horizon

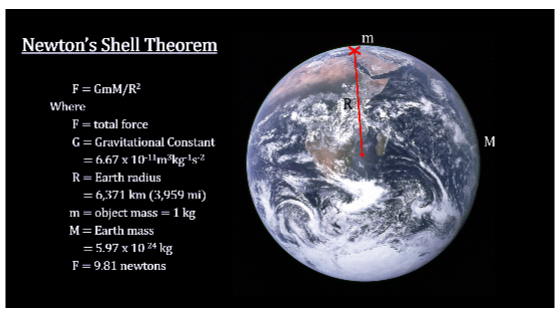


Then

$V_c = H R_h$
$R_h = V_c / H$
$= (3 \times 10^8 ms^{-1}) / 2.27 \times 10^{-18} s^{-1}$
$= 1.32 \times 10^{26}$ m $= 14 \times 10^9$ ly

Newton's Shell Theorem [Music: Tchaikovsky - Waltz from Sleeping Beauty]

We now turn our attention to the forces that are causing the Universe to expand. We'll start with Newton's gravitational equations. Back in the 1700s, Newton proved a theorem now named after him. It has two parts. The first is that an isotropic spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at its center.



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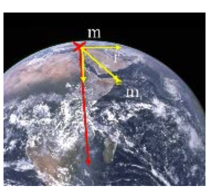
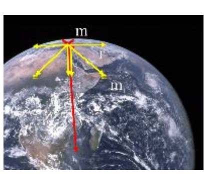
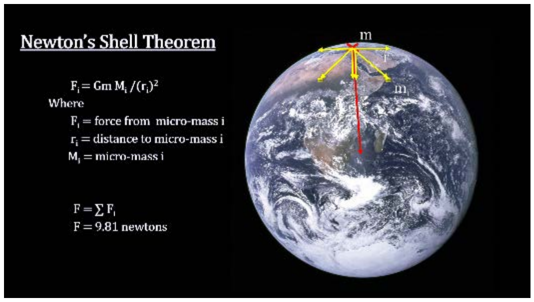


To illustrate how this works, consider a molecule of Earth off the line between a 1 kg object and the center of the Earth. It will have its own minuscule force on the object represented here as a vector with a magnitude (its length) and a direction. (The graphic is not to scale.) We can divide the vector into two parts, one pointing to the center of the Earth and the other perpendicular to it.



Because the Earth is isotropic, there is a corresponding molecule on the other side to the Earth with the same force magnitude, but in a different direction. Breaking this vector up as we did the first, we see that the vector component pointing towards the center will add to the equivalent component from the other molecule. The two perpendicular components will cancel each other out.

We get the total force on the object by summing all the minute forces from all the molecules in the planet. As you can see, the resulting vector points to the center.



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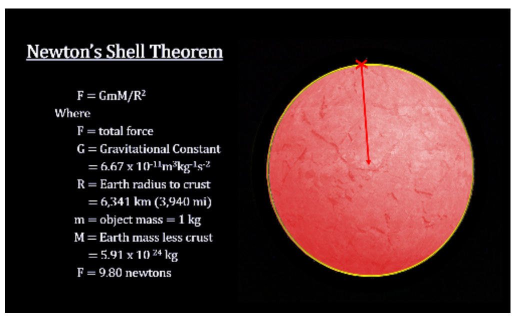
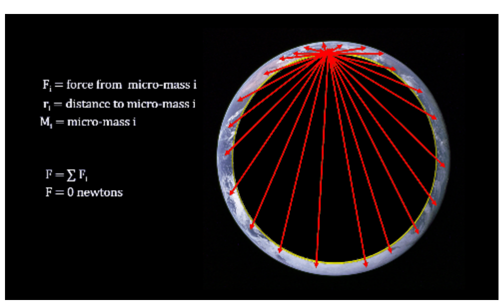
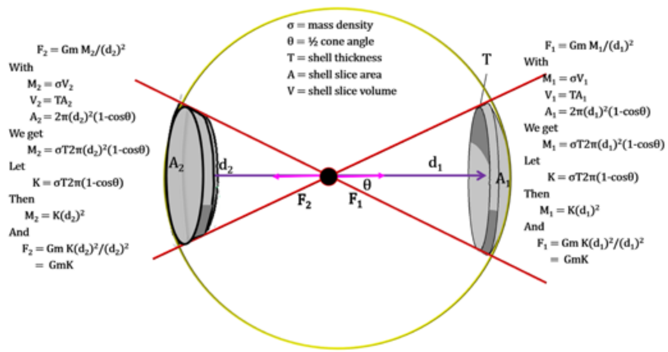
For part 2 of the theorem, we'll drill 30 km into the planet. From here we are on the Earth's Mantle and under the Earth's crust. Counting the oceans, around 1.05 % of the Earth's mass is now further from the center than our 1 kg object. We see that although the object is closer to the center, the overall force is just a little bit less due to the smaller mass.



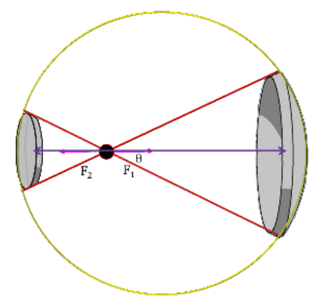
But what about all the matter in the Earth's crust that is now further away from the center than the object. The force on the object from this shell of matter will come from the sum of all the forces produced by all the molecules in the shell. What will the total be, and what would the total be if the object were elsewhere inside the sphere? Newton had to actually develop calculus to solve this problem. The remarkable result is that the sum total of all these forces is equal to zero. The shell mass has no impact on the object whatsoever. So now we have the second part of Newton's Shell Theorem. It states that a homogeneous spherically symmetric shell exerts no gravitational force on objects within the shell.



This is a key idea for our study of cosmology and it is not readily understood just why we would get a zero result anywhere inside the shell. A quick look at the geometry involved here helps. If we put our object at the center, and build a cone that intersects with the shell in two opposite directions, we can analyze the force on the object as it moves around inside the shell. At the center, all the forces cancel out.



Now move the object towards one side and away from the other. On the right side, the number of molecules is increasing by the square of the increasing distance as the force from each molecule is decreasing by the square of the increasing distance. The reverse is happening on the left side. The forces continue to cancel each other out. The total force remains equal to zero.



Escape Velocity [Music: Offenbach - Barcarolle (from Tales of Hoffman)]

The gravitational acceleration of an object on the surface, is always towards the center. But it is possible that the velocity of the object is away from the center. This happens when there is an initial velocity such as when a projectile is shot from a cannon.

To understand how the projectile will progress, we need to use the conservation of energy principle together with the total energy of the system at the time the projectile leaves the barrel of the cannon. (We'll ignore the effects of friction with the atmosphere in this analysis.) The initial velocity that enables the object to escape the Earth's gravitational pull is called the Escape Velocity. It is the velocity that makes the system's total energy equal to zero.

Energy Conservation

Let

- E_t = total energy
- E_k = kinetic energy
- E_p = potential energy
- m = mass of the object
- M = mass of the Earth
- R = radius of the Earth
- v = velocity
- v_e = escape velocity
- v_i = initial velocity
- a = acceleration

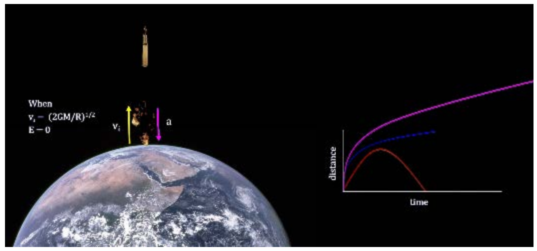
$$E_t = E_k + E_p$$
$$E_k = (1/2)mv^2$$
$$E_p = \text{-}GmM/r$$

When $E_k = E_p$, $E_t = 0$

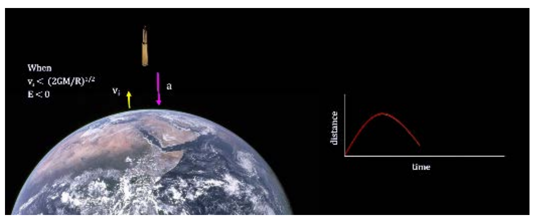
And

$$(1/2)mv^2 = GmM/r$$
$$v_e = (2GM/R)^{1/2}$$

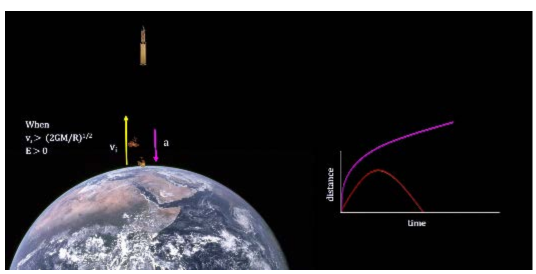
If the initial velocity is exactly equal to the escape velocity, the total energy is zero, and the object will slowly approach a velocity of zero but never quite reach it. It will never return to the Earth.



[Here's a bullet being fired vertically into the air. As is rises in the Earth's gravitational field, its gravitational energy will increase and its kinetic energy (the energy associated with the objects velocity) will necessarily decrease to keep the amount of energy constant.] In this example, the initial velocity is less than the escape velocity, so the total energy is negative, and the object will fall back to Earth.



If the initial velocity is greater than the escape velocity, the total energy is positive, and the object will go on with a nonzero velocity forever and never return to the Earth. It has escaped.

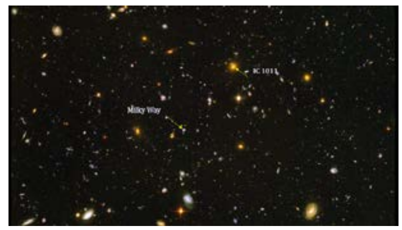


Newtonian Style Cosmology

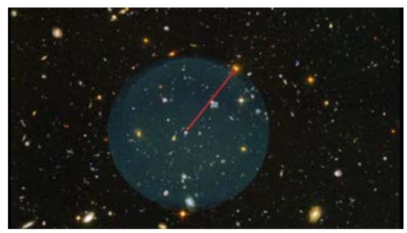
[Newton never applied his gravitation equations to the cosmos. There was no reason to. At the time, and for all time up to the 1920s, it was understood that the universe was vast and static. The stars looked fixed in the sky. But in the 1920s, with the advent of General Relativity, and Hubble's discovery, Alexander Friedmann, a Russian mathematician and physicist did. In fact, he did it with Newton's equations as the starting point and again with Einstein's General Relativity equations. We'll go over Newtonian Cosmology first.**]**

If we scale up our previous Earth bound bullet example, to galaxies in the Universe, we can apply Newton's equations for force, gravity, acceleration and his Shell Theorem to gauge how the Universe might evolve over time.

Picture a wide enough expanse of the Universe so that the Cosmological Principle holds true. We'll center our example on ourselves here in the Milky Way. We are at rest in our own reference frame. Consider a galaxy like IC 1101, a billion light years away. Our question comes down to "How will this galaxy move with respect to us".



Given that galaxies are made up of electrically neutral molecules, the only force at work here is gravity. If we build a sphere with us at the center and IC 1011 at the surface, we can calculate the gravitational force on and acceleration of IC 1011 from all the matter from all the galaxies within the sphere.



And we can use Newton's Shell Theorem to cancel out all the other gravitational forces in the Universe. We find that the mass density inside the sphere is all that matters.

Let

- R = radius of the sphere
- m = mass of IC 1011
- M = mass inside sphere
- V = volume of the sphere
- ρ = mass density inside sphere
- $\dot{R} = dR/dt$ (i.e. velocity)
- $\ddot{R} = d\dot{R}/dt$ (i.e. acceleration)
- F = force

We have

$$F = -GmM/R^2 = m\ddot{R}$$

$$\ddot{R} = -GM/R^2$$

$$\ddot{R}/R = -GM/R^3$$

Substituting $(4/3)\pi\rho R^3$ for M

We get

$$\ddot{R}/R = -(4/3)\pi G\rho$$

If we take a look at this equation for a minute, you can see its implications for cosmology. If the acceleration (R double dot) is zero, then the mass density of the Universe (roe) would have to be zero! In other words, the Universe cannot be static unless it is empty! The existence of matter in the Universe means that galaxies must not only be moving, they must be accelerating.

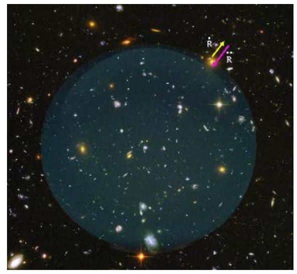
$$\ddot{R}/R = -(4/3)\pi G\rho$$

If R is a constant
Then $\dot{R}$ and $\ddot{R} = 0$

If $\dot{R}$ is a constant
Then $\ddot{R} = 0$

If $\ddot{R} = 0$
Then $\rho = 0$

The other thing to note is that the acceleration is always negative, meaning that it is in the direction of contraction. But we know from our examination of escape velocity, that initial conditions can have the Universe expanding even as that expansion is slowing down.



Looking at it from an energy point of view, Friedmann, using Newton's model, developed an equation, now named after him, which showed how the Universe would behave under various initial energy conditions.

Let

U = total energy per unit mass

E_k = kinetic energy per unit mass

E_p = potential energy per unit mass

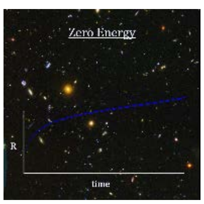
We have

$$E_k + E_p = U$$

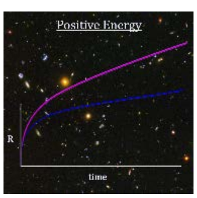
$$(1/2)\dot{R}^2 - GM/R = U$$

$$\dot{R}^2 = 2GM/R + 2U$$

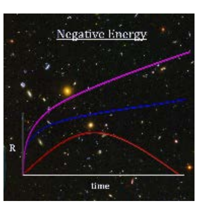
$$\left.\begin{array}{l}(\dot{R}/R)^2 = (8\pi G/3)\rho + 2U/R^2 \\ \text{Where} \\ \quad U \text{ is constant}\end{array}\right\} \text{Friedmann's Equation from Newton's gravitation}$$



The constant U in the equation represents the total energy per unit mass at the surface of the expanding sphere. There are three possibilities for this constant. It will be 0 if the kinetic energy is equal to the gravitational binding energy. In this case, an early rapid expansion will continue to slow as it approaches a steady volume, but never reaches it. This is like the bullet example's having the exact escape velocity.



It will be a positive number if the kinetic energy is large enough to overcome the gravitational binding energy. In this case, the Universe will expand forever. This is like the bullet example's having exceeded the escape velocity.



And it will be a negative number if the kinetic energy is insufficient to overcome the gravitational binding energy. In this case, the Universe will eventually collapse. This is like the bullet example's having less than the escape velocity.

Note that R dot over R is velocity over distance. This is the Hubble Constant. We see that it can vary with time. This means that the Hubble constant is not really constant. We call it the Hubble Parameter. The value we've measured is designated H-naught, and represents the value of the Hubble Parameter at the current time.

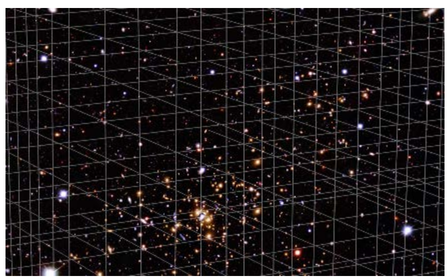
$$\dot{R}/R = V/D = H(t)$$

$$H(t)^2 = (8\,\pi G/3)\,\rho + 2U/R^2$$

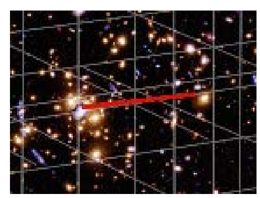
$$H_0 = H(t) \text{ where } t = \text{current time}$$

The Cosmic Scale Factor

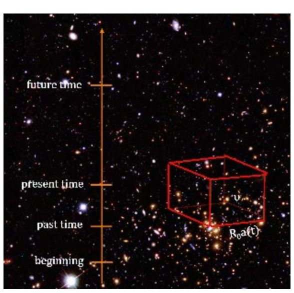
In order to more precisely analyze our expanding Universe, modern cosmology places a grid over our three dimensional space.

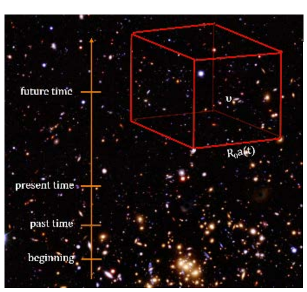


We treat the distance between two galaxies (R) as a constant. Then, we set the grid's scale factor 'a' equal to one at the present time, and vary it, to account for changes in distances over time instead of changing R.



Now consider a cube inclosing a volume of space containing some number of galaxies. With our scale factor approach, the amount of matter inside the volume remains the same as the volume increases or decreases. But the matter density goes down when the scale factor increases, and it goes up when the scale factor decrease.





Let

D = distance
V = velocity
A = acceleration
R_0 = current distance
$a(t)$ = scale factor
υ = mass in unit volume
ρ = mass density

Then

$D = R_0\, a(t)$
$V = R_0\, \dot{a}$
$A = R_0\, \ddot{a}$
$\rho = \upsilon/a^3$

Sticking with Newton's model and incorporating the cosmic scale factor, we can rewrite the Friedmann equation. We see that the scale factor 'a' is the only variable. In other words, the history of the Universe comes down to the history of the scale factor. And the history of the scale factor depends completely the contents of the Universe and how that content effects the space it exists in.

With the scale factor 'a'

$$(\dot{R}/R)^2 = (8\pi G/3)\rho + 2U/R^2$$

Becomes

$$(\dot{a}/a)^2 = (8\pi G/3)\rho + 2U/R_0^2 a^2$$

And

$$(\dot{a}/a)^2 = (8\pi G\upsilon)/3a^3 + (2U)/R_0^2 a^2$$

$$(\dot{a}/a)^2 = K_M/a^3 + K_E/a^2$$

Where

$$K_M = (8/3)\pi G \upsilon$$

(a constant based on total matter)

And

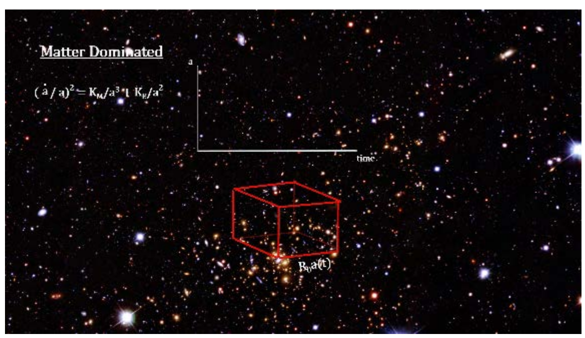
$$K_E = 2U/R_0^2$$

(a constant based on total energy)

[Newtonian Matter Domination

Let's examine what this version of the Friedmann equation says about the evolution of the Universe under differing circumstances. In particular, we'll cover 3 cases. One is for a matter dominated Universe (one that's mostly filled with matter). The second is for a radiation dominated Universe (one that's mostly filled with photons). And the third is a mixed Universe with both.

The version we've been working on, up to this point, reflects a matter dominated Universe because it was derived using matter density (roe).



If the total energy of the universe is zero, Friedmann's equation can be simplified and solved to show how the scale factor changes with time. We see that the Universe would grow proportional to time raised to the 2/3 power. That's the cube root of time squared. Galaxy velocities will slow down, but never reach zero.

$(\dot{a}/a)^2 = K_M/a^3 + K_E/a^2$

If $a << 1$

Then $K_E/a^2 << K_M/a^3$

And $(\dot{a}/a)^2 = K_M/a^3$

$\dot{a} = da/dt = (K_M/a)^{1/2}$

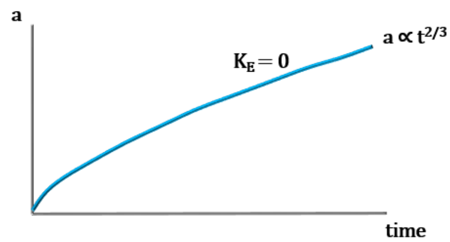
$dt/da = (a/K_M)^{1/2}$

$t = (2/3)K_M^{-1/2}a^{3/2}$

Solving for 'a' as a function of t

$a = kt^{2/3}$

Where $k = (9K_M/4)^{1/3}$



If the energy of the Universe is not zero, the energy term takes over as the Universe expands. Again we can simplify and solve for how the scale factor changes with time. If the kinetic energy of the matter in the universe is greater than the binding gravitational energy, the total energy of the universe is positive, and galaxies move off with a uniform velocity.

$(\dot{a}/a)^2 = K_M/a^3 + K_E/a^2$

If $a >> 1$

Then $K_E/a^2 >> K_M/a^3$

And $(\dot{a}/a)^2 = K_E/a^2$

$\dot{a}^2 = K_E$

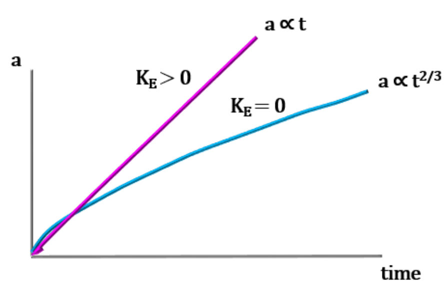
$\dot{a} = K_E^{1/2}$

Solving for 'a' as a function of t

$a = kt$

With

$k = K_E^{1/2}$



If the kinetic energy of the matter is less than the binding gravitational energy, the total is negative, and the universe expands for a time, reaches a maximum size and then the process reverses.

If $K_E < 0$

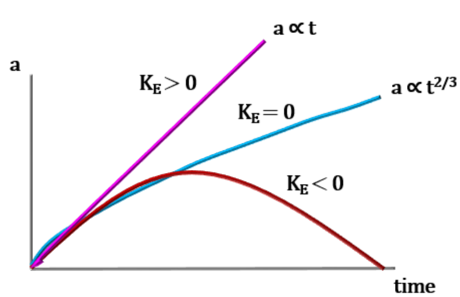
Then, while a is small

$a \propto t^{2/3}$

As a increases

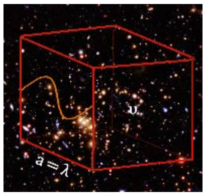
$(\dot{a}/a)^2 = K_M/a^3 - |K_E|/a^2 \Rightarrow 0$

And expansion reverses



Newtonian Radiation Domination

A radiation dominated Universe is composed mostly of photons. Unlike matter that moves through space, photons are attached to the space they propagate through. So an expanding space will impact photons in a way that does not affect matter.



Here's a cubic volume of space with a photon inside. The photon's wavelength (λ) is equal to the length of the cube (a). Its energy is equal to Planck's constant time the speed of light divided by the wavelength (E = hc/λ).

Let

- υ = energy in unit volume
- ρ = energy density
- λ = wavelength
- h = Planck's constant
- c = speed of light

Then

$$E = hc/\lambda$$
$$E = hc/a$$
$$\rho = hc/a^4$$

And

$$(\dot{a}/a)^2 = 8\pi G hc\upsilon/3a^4 + K_E/a^2$$
$$(\dot{a}/a)^2 = K/a^4 + K_E/a^2$$

Where $K = 8\pi G hc\upsilon/3$

As the wavelength increases with an increase in the scale factor, the energy decreases - unlike matter where it remained constant. The reduction in energy density then becomes one over the scale factor to the 4th power instead of cubed like we had with matter. This gives us a different expansion over time than we got with matter. The lost energy is going into the kinetic energy of the space expansion.

Matter Dominated

$$(\dot{a}/a)^2 = K_M/a^3 + K_E/a^2$$

If $a << 1$

Then $K_E/a^2 << K_M/a^3$

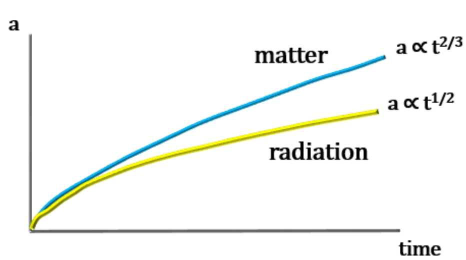
And $(\dot{a}/a)^2 = K_M/a^3$

$$\dot{a} = da/dt = (K_M/a)^{1/2}$$
$$dt/da = (a/K_M)^{1/2}$$
$$t = (2/3)K_M^{-1/2}a^{3/2}$$

Solving for 'a' as a function of t

$$a = kt^{2/3}$$

Where $k = (9K_M/4)^{1/3}$



Radiation Dominated

$$(\dot{a}/a)^2 = K/a^4 + K_E/a^2$$

If $a << 1$

Then $K_E/a^2 << K/a^4$

And $(\dot{a}/a)^2 = K/a^4$

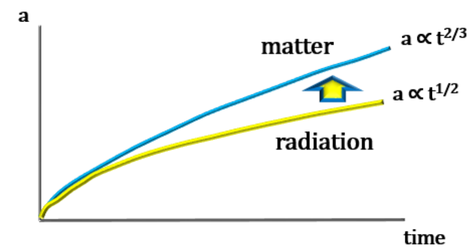
$$\dot{a} = da/dt = K/a$$
$$dt/da = a/K$$
$$t = (K/2)^{-1}a^2$$

Solving for 'a' as a function of t

$$a = kt^{1/2}$$

Where $k = (2K)^{1/2}$

We can conclude that, as the contents of the Universe changed from radiation to matter, the cosmic scale factor underwent a transition from a slower expansion, to a faster expansion. This is Newtonian style cosmology.



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Friedmann Equations

Understanding the evolution of the Universe is what cosmology is all about. Up to this point, we've been using the Newtonian equations. For a full picture we need to use Einstein's general theory of relativity that includes mass-energy and pressure. Plus we need to consider the curvature of space-time given its mass-energy contents.

Einstein Field Equations

$$G_{\mu\upsilon} = \frac{8\pi G}{c^4} T_{\mu\upsilon}$$

Robertson-Walker Metrics

Flat space:

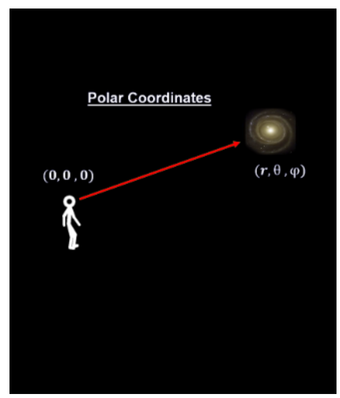
$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]$$

Positive curvature:

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + R_0^2(\sin^2(r/R_0)(d\theta^2 + \sin^2\theta d\varphi^2)]$$

Negative curvature:

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + R_0^2(\sinh^2(r/R_0)(d\theta^2 + \sinh^2\theta d\varphi^2)]$$



Here's the equation Friedmann developed from this starting point.

From General Relativity

$$(\dot{a}/a)^2 = 8\pi G\varepsilon(t)/3c^2 - \kappa c^2/R_0^2 a(t)^2$$

From Newtonian mechanics

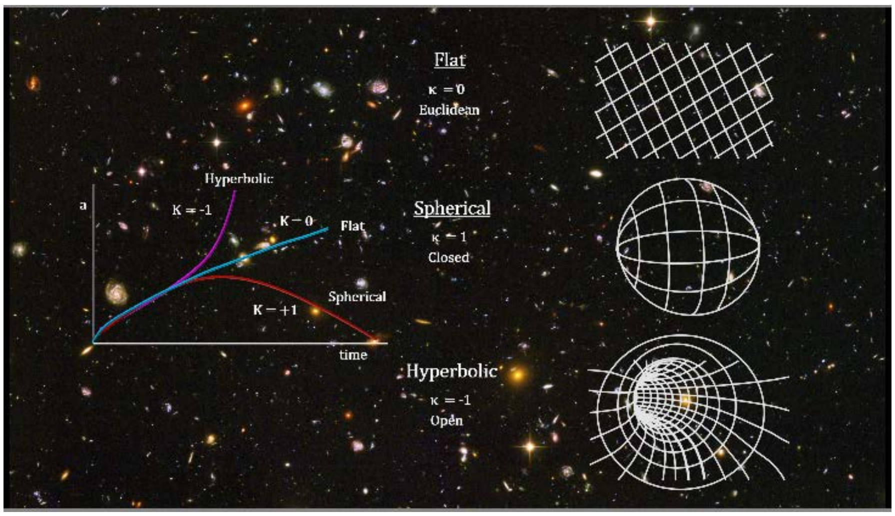
$$(\dot{a}/a)^2 = 8\pi G\rho(t)/3 + (2U)/R_0^2 a(t)^2$$

It is quite similar to the Newtonian version with two key differences. First, the mass-density is replaced by the energy-density. And second, the total energy is replaced by the radius of curvature and the curvature constant that equals -1, 0, or +1 and tells us which metric to use depending on the nature of the curvature.

where replaces	$\varepsilon(t)/c^2$ $\rho(t)$	energy density mass density
and replaces	κc^2 2U	curvature [-1, 0, 1] total energy
and	R_0	becomes the radius of curvature

For an isotropic homogeneous universe this says we must exist in one of these three possible universes:

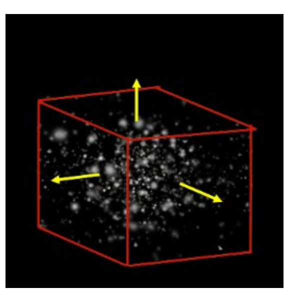
- If it's flat, the Universe will expand forever at an ever decreasing rate.
- If it's spherical, it is closed and will eventually collapse back into a big crunch.
- If it's hyperbolic, it is open and will expand forever at an increasing rate.



Equation of State [Music: Bach - Double Violin Concerto]

To take into consideration the impact of Pressure – the other key component of Einstein's Field Equations, we need to consider the relationship between pressure and energy density. When we change the amount of energy in a box, the pressure on the walls changes. In a slowly changing volume, the constant of proportionality is called w and varies depending on the nature of the contents of the box. In Physics, this is called the equation of state.

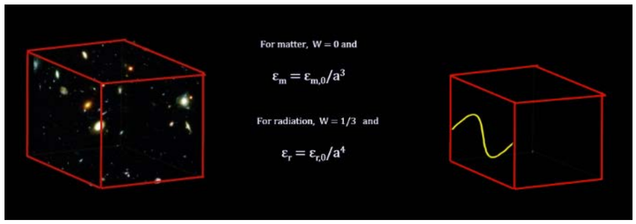
$$
\begin{aligned}
E &= \varepsilon V = \varepsilon a^3 \\
P &= w\varepsilon \\
dE &= -PdV \\
dE &= \varepsilon dV + Vd\varepsilon = -PdV \\
Vd\varepsilon &= -(P+\varepsilon)dV \\
&= -(w\varepsilon+\varepsilon)dV \\
d\varepsilon/\varepsilon &= -(1+w)dV/V \\
\varepsilon &= \varepsilon_0/V^{(1+w)} \\
\varepsilon &= \varepsilon_0/a^{3(1+w)}
\end{aligned}
$$



Where

- P = pressur
- ε = energy density
- ε_0 = energy density now
- w = constant of proportionality
- E = energy
- V = volume
- a = cosmic scale factor

For matter that is not actually moving inside the box, the pressure is zero, so w is 0. For radiation, it can be shown that w equals 1/3 where the number 3 comes from the number of spatial dimensions. This gives us the same relationship between energy density and the scale factor that we had with the Newtonian version.



With this and the Friedmann Equation, we can now calculate the history of the scale factor and thereby the history of the Universe if we can determine its radiation energy density, matter energy density and curvature. By the same token, we can calculate the energy density in the Universe, if somehow we can quantify the scale factor.

$$(\dot{a}/a)^2 = 8\pi G\varepsilon(t)/3c^2 - \kappa c^2/R_0^2 a(t)^2$$

$$H^2 = \underbrace{(8\pi G\varepsilon_{r,0}/3c^2)/a^4}_{\text{radiation}} + \underbrace{(8\pi G\varepsilon_{m,0}/3c^2)/a^3}_{\text{matter}} + \underbrace{(-\kappa c^2/R_0^2)/a^2}_{\text{curvature}}$$

Critical Density

We define the Critical Density of the Universe as the density that would give us flat space. Any more than this and we have a closed Universe. Any less than this and we have an open Universe. The flat matter dominated version is called the Einstein – de Sitter Universe after the scientists that developed the model. This critical mass-density comes to around is 5 protons per cubic meter [8.7 x 10^{-27} kgm^{-3}]. The actual density of interstellar space is on the average of about 1 proton per cubic centimeter. That's a million times denser than the critical density. But much of the Universe is made up of vast voids with far less than this, so 5 protons per cubic meter could be the number we actually have.

$$(\dot{a}/a)^2 = 8\pi G\varepsilon(t)/3c^2 - \kappa c^2/R_0^2 a(t)^2$$

If flat, then $\kappa = 0$ and

$$(\dot{a}/a)^2 = H(t)^2 = 8\pi G\varepsilon(t)/3c^2$$

And

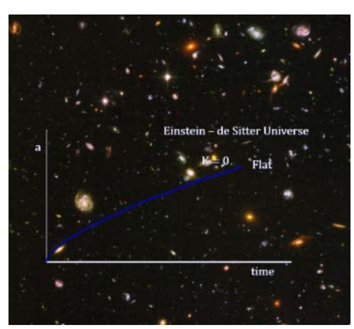
$\varepsilon_c(t) = 3c^2H^2(t)/8\pi G = 4{,}870 \text{ MeV/m}^3$

Where

$\varepsilon_c(t)$ is the critical energy density

In mass-density units at the current time we get

$\rho_{c,0} = \varepsilon_{c,0}/c^2 = 8.7 \times 10^{-27} \text{ kg/m}^3$



Cosmologists like to work mostly with ratios. In this case, we have the ratio of the energy-density over the critical density called the density parameter omega. Current measurements have it at very close to 1 [0.995 < Ω < 1.005]. It is the sum of all forms of energy that fill the Universe. At this point in our analysis, we have three components that add up to 1: radiation, mass, and curvature.

$$\Omega(t) \equiv \varepsilon(t)/\varepsilon_c(t)$$

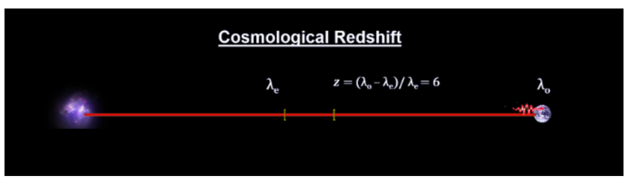
$$0.995 < \Omega_0 < 1.005$$

$$\Omega = \Omega_r + \Omega_m + \Omega_c = 1$$

Where
- Ω_r = density parameter for radiation
- Ω_m = density parameter for matter
- Ω_c = density parameter for curvature

Cosmological Red Shift

When we observe light from distant galaxies, we are seeing the light from the stars in those galaxies. And that light has absorption lines. The same lines measured in a lab give us the wavelength of the light at the time it was emitted. What we observe, is the wavelength stretched over the time it took to get here [link to redshift]. We define redshift z as the difference between the two divided by the wavelength emitted. In this hypothetical example, we have an object with a redshift equal to 6.



Once a model for the change in the cosmic scale factor over time is specified, redshift gives us a great deal of information. For now, we'll assume a flat, matter dominated Einstein de Sitter universe. This will only get us part of the way to the actual numbers, but it helps illustrate the key role redshift plays in cosmology.

Matter Dominated flat model

$$H_0 = 2.2 \times 10^{-18} s^{-1}$$
$$\Omega_r = 0$$
$$\Omega_m = 1$$
$$a(t) = \left({}^{t}/_{t_0}\right)^{2/3}$$

First, redshift gives us an object's receding velocity. With our model, we have the object moving away at 6 times the speed of light. Redshift also gives us the actual cosmic scale factor at the time the light was emitted. When the light we see from this object started, the Universe was a little over a tenth of its current size.

Z Scale Factor & Time

$$V_r = zc = 6c$$
$$a(t_e) = 1/(z+1) = 0.14$$

It gives us the age of the universe at the time the light was emitted, and the amount of time the light was traveling. We're seeing it as it looked almost 8 billion years ago when our Universe was only 2 billion years old.

$t_0 = 2/3H_0 \quad = 9.6$ billion years

$t_e = t_0 a(t_e)^{2/3} \quad = 1.9$ billion years

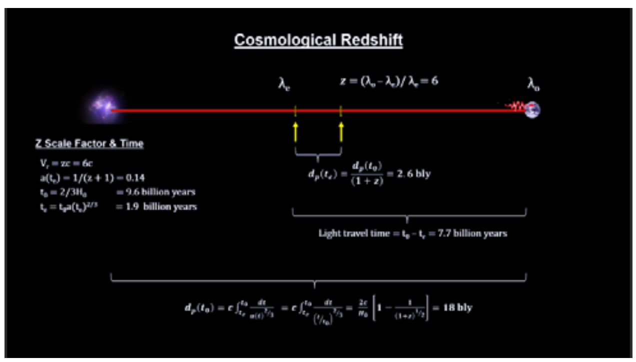
Light travel time $= t_0 - t_e = 7.7$ billion years

Redshift gives us the distance to the object at the current time. And it gives us the distance to the object at the time the light was emitted.

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)^{2/3}} = c\int_{t_e}^{t_0} \frac{dt}{\left({}^{t}/_{t_0}\right)^{2/3}} = \frac{2c}{H_0}\left[1 - \frac{1}{(1+z)^{1/2}}\right] = 18 \text{ bly}$$

$$d_p(t_e) = \frac{d_p(t_0)}{(1+z)} = 2.6 \text{ bly}$$

You can see why astronomers rely so heavily on redshift measurements. Next, we'll use it extensively to count galaxies.



Counting Galaxies

We now turn our attention to some cosmological observations.

The way cosmologists judge any given model for the Universe, is to compare the model's predicted outcomes with what we actually observe. You may recall from our video book on General Relativity, that curved space has different volume implications than we have for Euclidean flat space. [Riemannian Curvature] So one way to determine if the Universe is flat, spherical or hyperbolic is to count galaxies at different redshifts (i.e. different 3distances).

Galaxy Counts

Let

r_1 = distance to inner shell wall

r_2 = distance to outer shell wall

V_1 = volume of the inner sphere

V_2 = volume of the outer sphere

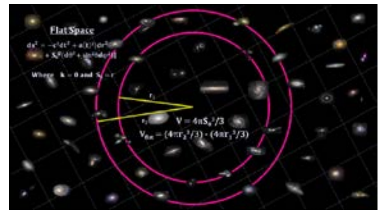
V = volume of shell

N = number of galaxies within shell

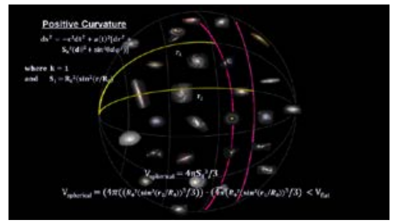
We have

$N \propto V$

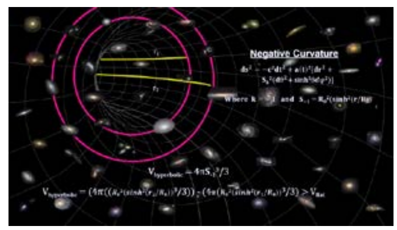
If we look out into a flat universe, we would see the number of galaxies vary with the volume.



If we look out into a spherical space, we would see the number of galaxies increase more slowly than for flat space, reach a maximum, and then come back down. [At the maximum distance, we would find just one filling the whole sky.]



And if we look out into a hyperbolic space, we would see the number of galaxies increase dramatically faster than for flat space.

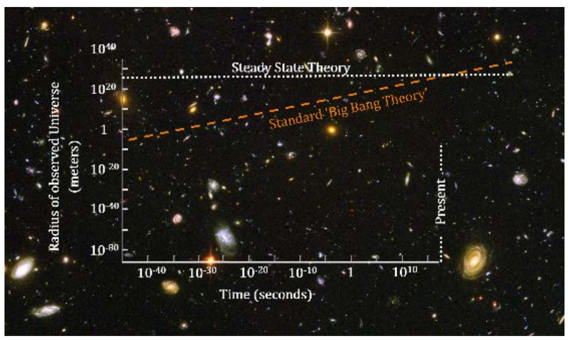


With our modern technologies, we can see galaxy populations out to around 10 billion light years. [That would be a redshift of 0.726.] And as far as we can see, the number of galaxies increases according to the flat space model. Observations of galaxy diameters and luminosity distances also show a flat Universe. Large scale space looks completely flat! [This is consistent with the measurements that show that the 'density parameter' Ω is very close to 1.] But, if the Universe is large enough, say with a radius of curvature at around 200 billion light years, it is possible for it to look flat to us examining such a small part of it. So it is still possible that we live in a 3-dimentional sphere with a huge radius of curvature. But for the rest of our study of the benchmark model, we'll assume we exist in flat space.

Big Bang vs Steady State Cosmology [Music: Rimsky-Korsakov - Scheherazade]

By the early 1960s, we had a consistent theory for how the Universe scaled over time, with three basic curvature models. Observations of galaxy counts at different distances indicated that the flat model was the best fit. And Type 1a Supernovae studies gave us a good reading on the value of the Hubble Parameter at the current time.

But the Big Bang theory was challenged by a Steady-State cosmology that held that the cosmological principle was true for all observers and for all time. Galaxy redshift was explained as photons loosing energy to space rather than space expanding. Plus, the supporters argued that a big bang would have left a trace that should be detectable, but nothing had been found.



That all changed when the Cosmic Microwave Background (CMB) radiation was discovered in the mid-60s. That's where we'll begin part 2 of "How old is it – The Universe"