



How Old is the Solar System

{Abstract: *In this chapter we cover the overall formation of the Solar System – the phases it went through and the time it took. We start with the theory of planetary formation in general and then home in on our planetary system with its inner rocky and outer gas planets. Then we cover the contents of our circumstellar disk including all the elements we find here on Earth. We focus on the dust because it represents the initial building blocks for planets. We then cover how this dust can grow into planetesimals that grow into asteroid sized objects that in turn grow into planet sized objects. We see that, once objects get large enough, they sweep up all the matter in their orbits. This process ends when the Sun’s solar wind clears out the remaining gas and dust. We show how the activity was extremely chaotic, but it winds up with 9 planets – four inner rocky planets and 5 outer water rich planets with a “Snow Line” separating them. We then cover dating meteorites like Canyon Diablo and Allende. We take a deep dive into the lead-lead dating method because it is what tells us the age of the Sun and the Solar System. We conclude with a review of the Solar Systems development milestones.* }

Planetary Formation Theory Introduction

In ‘How old are stars’ we found that the Sun will burn hydrogen for 10 billion years, but because the Sun is a ‘Field Star’, we could not determine how long ago the hydrogen burning started. In ‘How old is the earth-moon system’ we found that the oldest solid Earth and Moon material was 4.3 billion years old, but we could not determine the age of the Earth from its original start because the Giant Impact turned the mantel to magma. In this chapter we’ll cover the age of the Solar System itself which will give us both the age of the Sun and the age of the Earth. To get a handle on the age of the Solar System, we’ll need to review planetary formation theory. Any such theory would need to explain our solar system as we see it today.

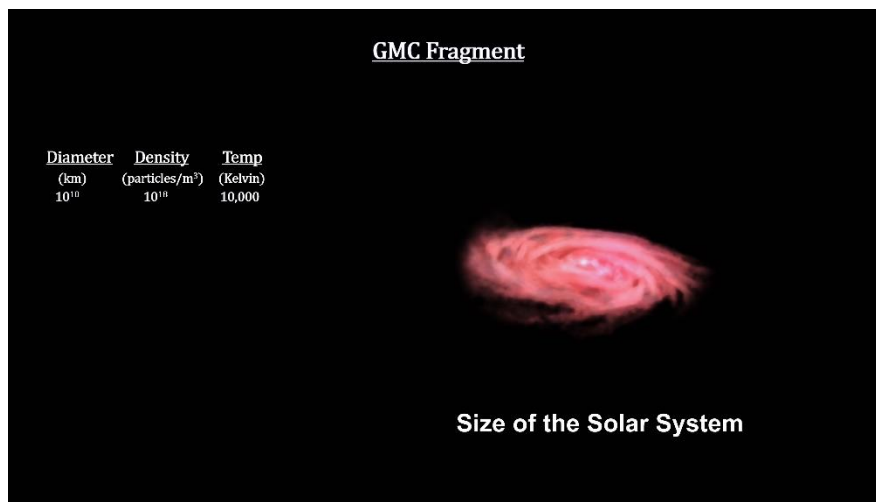
We’ll start with a look at some of the key characteristics of our system. We have the sun at the center with four relatively small rocky planets in the inner solar system and four much larger gaseous planets in the outer solar system. They are all in nearly circular orbits and they are all in the same orbital plane. None of the planets are orbiting outside the plane like some comets, asteroids, and dwarf planets do. Our theory of planetary formation will need to explain these facts.



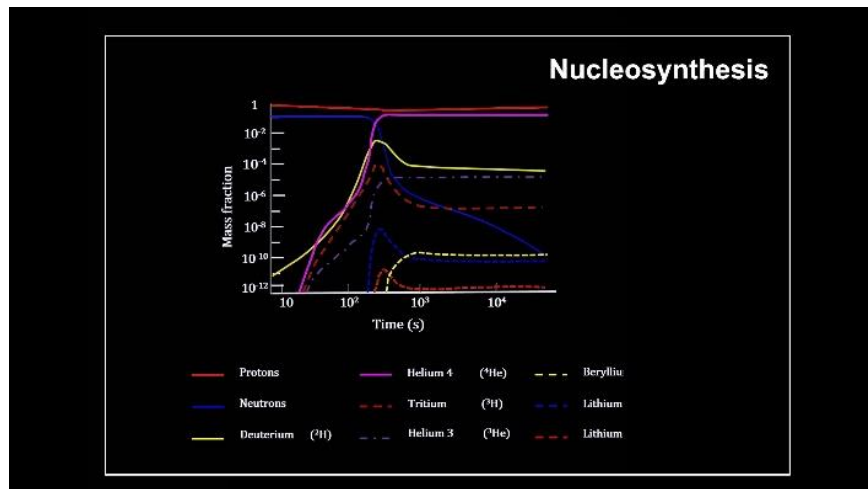


Circumstellar Disk Contents

In our “How Old Are Stars” segment, we covered how the circumstellar disk feeds the central object leading to the formation of protostars, T-Tari stars and eventually fully formed helium fusion burning stars. The process from the beginning of the protostar phase to a fully-fledged star is estimated to take from one to two hundred million years. During this period, the debris in the disk is forming planets.

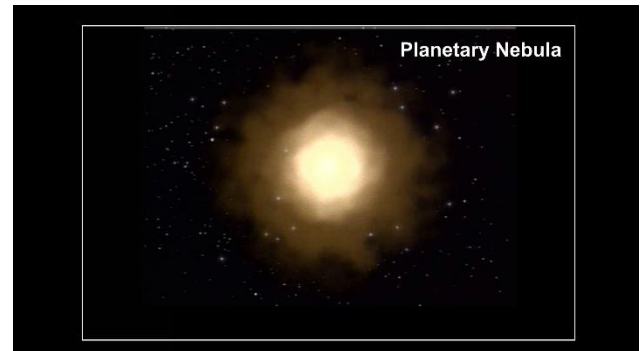


In the Nucleosynthesis segment of the ‘Lambda Cold Dark Matter Big Bang Theory’ chapter we covered the content of clouds like these for the first generation of stars. At that time, it was limited to hydrogen, some helium and just traces of beryllium and lithium.



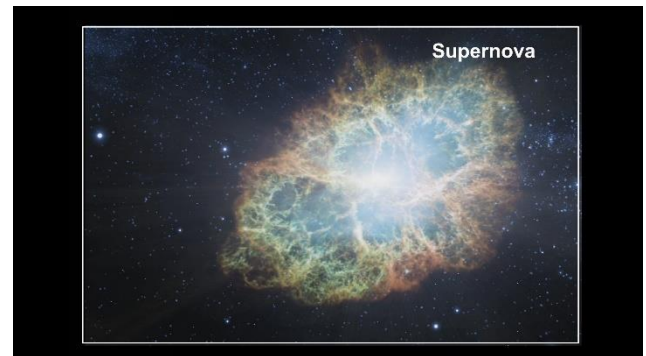


Planetary nebulae form when normal stars run out of hydrogen. The violent process created pressures and temperatures large enough to fuse hydrogen and helium into heavier elements and eject them into the interstellar medium from which giant molecular clouds like this are formed. These include carbon, nitrogen, and oxygen along with even smaller amounts of heavier elements like silicon, sulfur and iron. But this kind of star transition does not have energy to fuse enough protons to create atoms larger than iron.

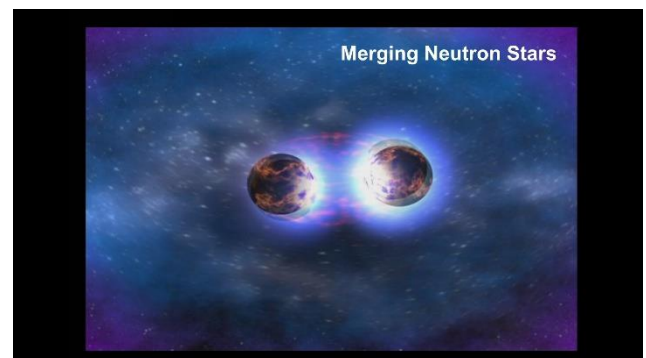


Supernovae explosions occur when super massive stars run out of hydrogen. These seed even heavier elements into the interstellar medium like lead, zirconium, silver, tungsten and gold. But even at these extreme energies, it is unlikely that a supernova could produce elements larger than lead. This is because the repulsive force of like charges is so strong – the Coulomb Force creates the Coulomb barrier. You may recall from our last chapter on ‘How Old Are Stars’, that a proton in our Sun can collide a trillion times a second with

other protons and still not fuse for a billion years.

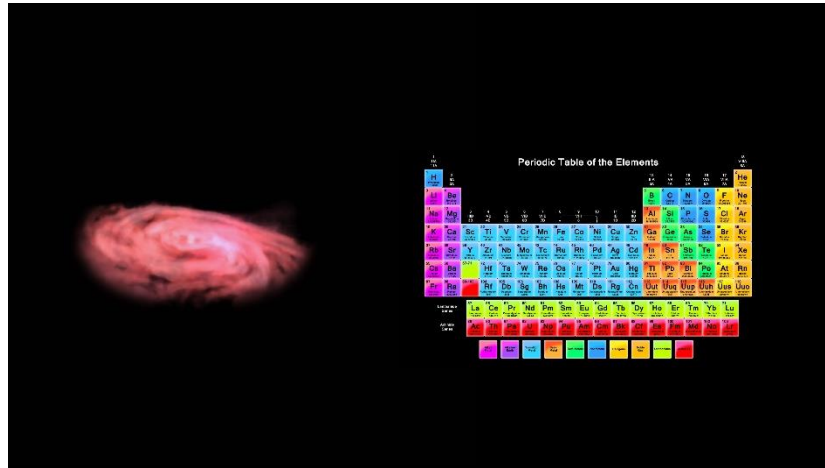


But neutrons have no charge and their fusing has no such barrier. It has long been theorized that the heaviest elements like thorium, protactinium and uranium were created by neutron star mergers. In 2017, using large laser interferometers that we covered in the ‘Gravitational Waves’ segment of the ‘How fast is it’ video book, just such a merger was detected. Now known as a kilonova, it's understood that neutron star mergers are the origin of the majority of all the heaviest elements found throughout the Universe.





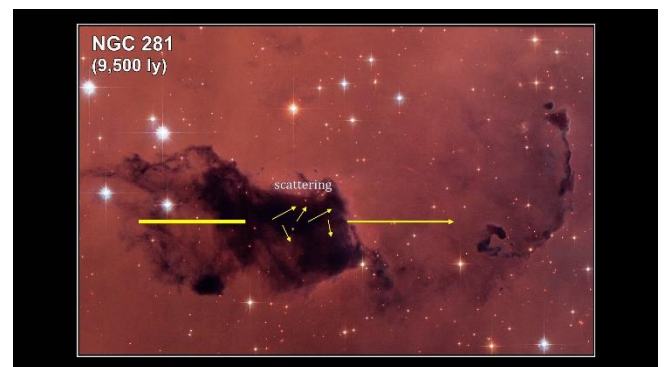
For our collapsing cloud, we know that it contained all 94 natural elements because we find them here on Earth.



Circumstellar Disk Dust

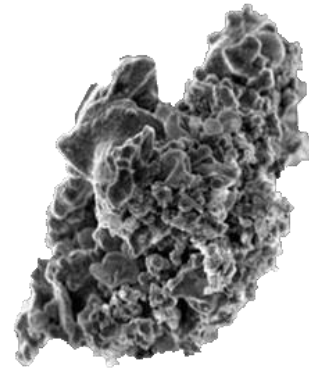
99% of the mass of the circumstellar disk is in the form of gas with just 1% in the form of dust. The solid dust has little effect on the star formation, but it's key to planetary formation. Dust is the only solid grains available for growing planets. Dust itself cannot be formed directly from purely gaseous material at the low densities found in interstellar molecular clouds. [For a solid to condense, the gas density must be high enough to allow a few atoms to collide and stick together long enough to radiate away their energy and cool.] Instead solid grains are known to form in planetary nebulae, supernovae and in the outer atmospheres of cool supergiant stars. The dust in the interstellar medium extinguishes light from stars via absorption and scattering.

The scattering leads to emissions of their own. Comparing dusty clouds to non-dusty clouds using spectral absorption and emission lines, shows that almost all the iron, magnesium, silicon, much of the carbon, and some of the oxygen and nitrogen are contained in the dust. This makeup is similar to terrestrial amorphous non-crystalline rocks. If the temperature permits, they are surrounded by a mantle of water ice. [In fact, water (H₂O) is abundant in the universe because there is a lot of hydrogen (74%) and a lot of oxygen (1%).]

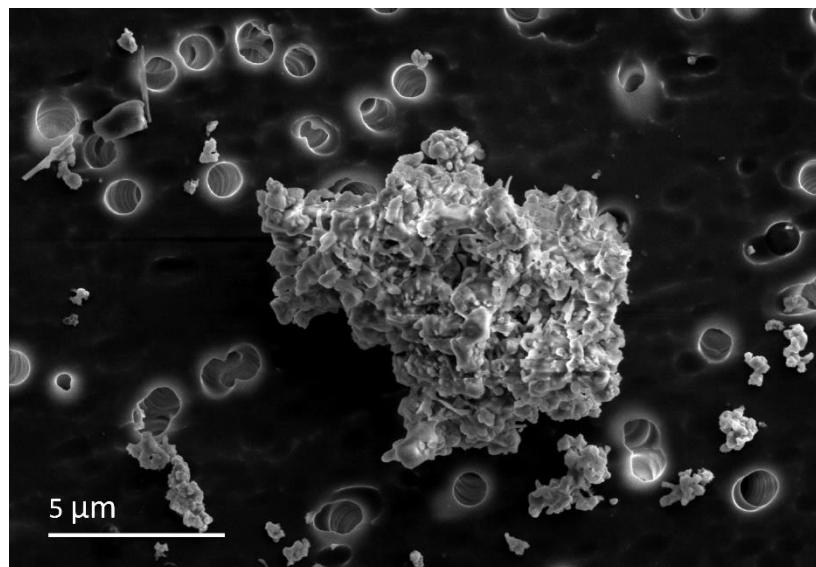




The original dust grains in the cloud are no longer available for direct observation. But to this day, there are similar objects in our solar system called interplanetary dust particles (IDPs for short). They are being collected in the thermosphere by the International Space Station. Here's an electron microscope view of one of them. It's 10 micrometers in length. That's around 100 times larger than interstellar dust.



NASA uses high flying aircraft to collect dust at high altitudes before it gets close enough to the surface to mix with Earth elements. In 2018, a team of scientists from the University of Hawaii 'i at Manoa, examined this dust with electron microscopes. They mapped the element distributions and discovered that these glassy grains are made up of subgrains that aggregated together prior to the formation of the comet the IDP came from. These represented samples of the early interstellar dust.



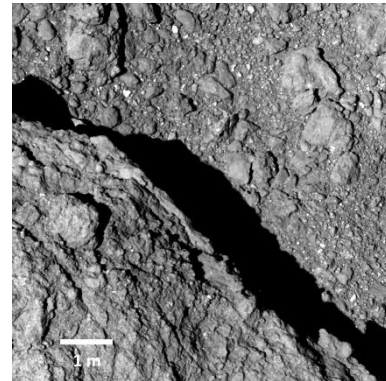


Planetesimals

Dust is the base material for planet formation. The dust grains run into each other and stick, forming larger and larger grains, mixing compounds and eventually forming mineral rich pebble sized objects that grow to bolder size. The near Earth asteroid 2015 TC25 is an example of an object this size. With a 4 meter diameter (that's 13 feet), it's one of the smallest asteroids ever detected. [1.4×10^4 kg]



The process continues to grow the rocks into rubble heaps large enough for a little gravity to hold them together. Ryugu is an example of an object this size. It's 1 km wide and weighs in at just under a half a trillion kg. Japan landed rovers on this asteroid in 2018. You can see the rubble nature of the object with this picture taken from the surface.

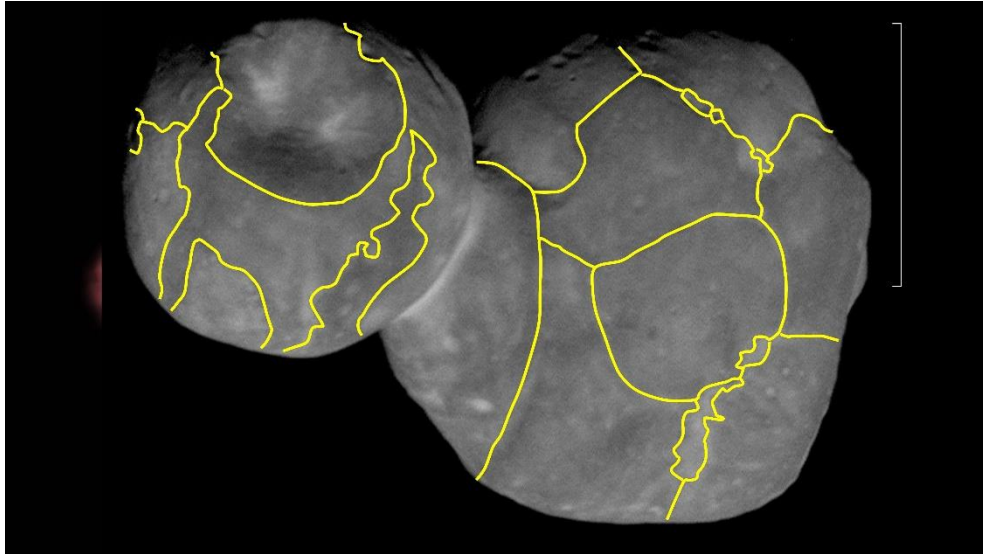


By the time enough matter has accumulated into objects like these, we have what astronomers call "planetesimals." These can extend from several to hundreds of kilometers in diameter. Comet 67P, visited by the Rosetta mission in 2014, is thought to be a combination of two planetesimals that bound together in a slow speed collision. Their combined mass is just under 10 trillion kg. [The larger lobe is about $4.1 \times 3.3 \times 1.8$ kms (that's $2.5 \times 2.1 \times 1.1$ miles) and the smaller one is $2.6 \times 2.3 \times 1.8$ km (that's $1.6 \times 1.4 \times 1.1$ mi).]

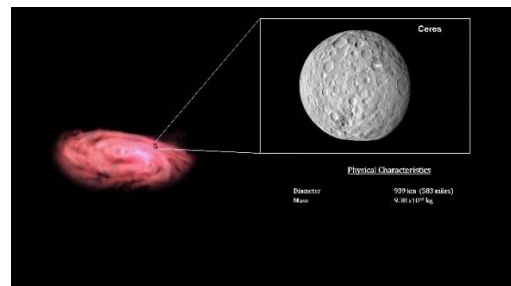




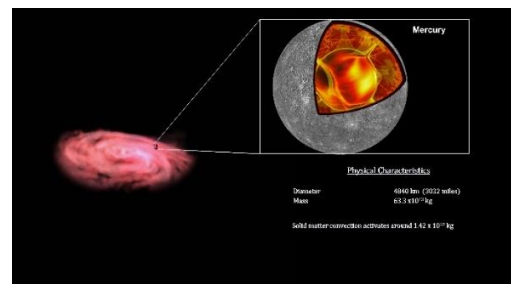
This is Arrokoth. It was discovered in 2014 out in the Kuiper Belt by the New Horizons Search Team using the Hubble Space Telescope. Like P67, it has two lobes that collided slowly, this object is 36 km across (that's 22 miles), and over a hundred times more massive. It's considered a minor planet like Pluto. A close examination of the surface shows lighter lines separating sections of each lobe. These indicate that Arrokoth was built piece by piece by the coalescing of over a dozen smaller planetesimals.



By the time this collide and merge process creates objects with enough mass to produce a gravitational strength that exceeds the structural strength of the rocks, the object is forced into a spherical shape. Ceres is a good example of this.

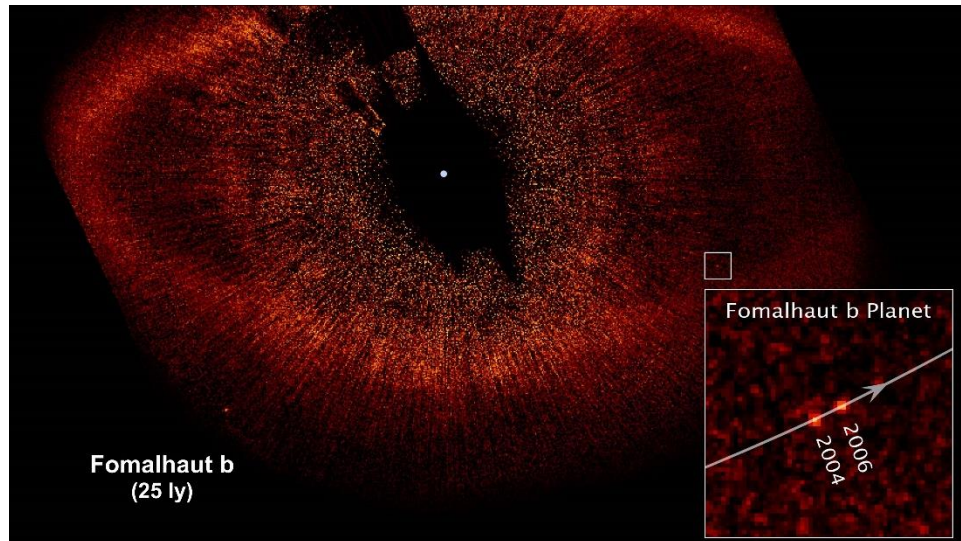


And once the mass reaches around 14 billion trillion kg (1.42×10^{22}), solid matter convection activates. The temperatures and pressures inside the object liquify matter and the core becomes molten. Mercury, our smallest planet, is a good example of this.





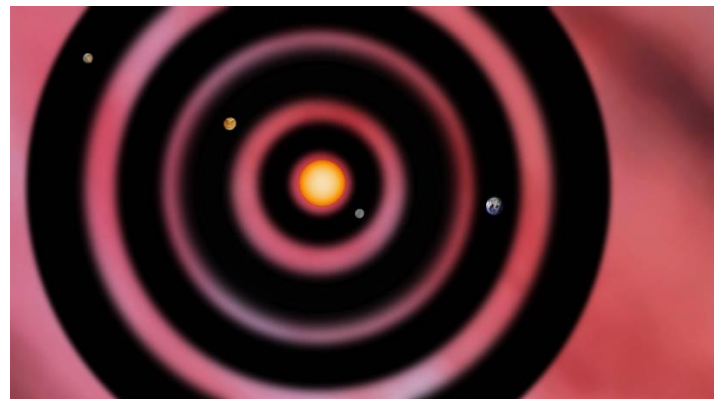
The star Fomalhaut is a good example of this process. Its circumstellar disk morphed into a protoplanetary disk with at least one object large enough to be considered a planet – Fomalhaut B.



Sweeping Out Orbits

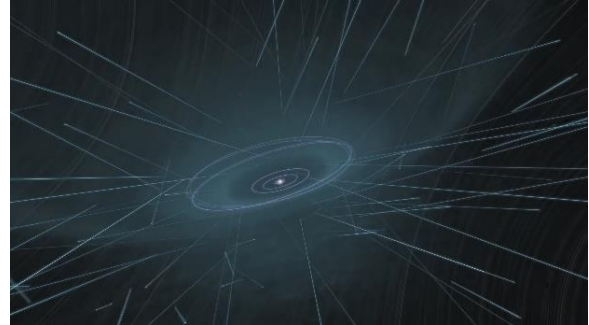
Over time, these larger objects continue to grow by accumulating matter from the disk. We find that in each region, each orbit, each distance from the Sun, everything coalesces into one massive object. These larger objects sweep out the remaining debris in their orbits. This is a defining characteristic for planets. All the little deviations averaged out as the smaller particles with varying elliptical orbits combined. This explains orbits being nearly circular and all in the same plane (co-planar). But the actual process is very chaotic – not as simple and straight forward as this illustration.

The process of accreting matter in the disk into larger objects came to an end when the Sun ignited as a main sequence star, and its strong solar winds blow away all the remaining loose material. Computer model estimates for how long this process might take range from 100 million to 200 million years. That's around the same timeframe it took the Sun as a protostar to reach steady state on the main sequence.





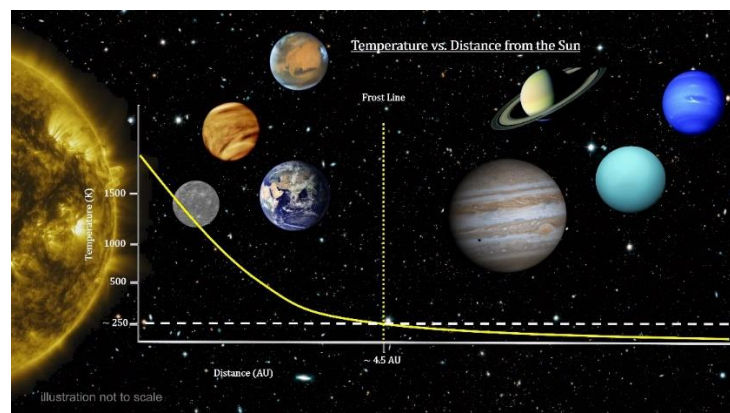
Here's a computer simulation created by Caltech that illustrates this chaos. Planets interact with the rotating disk and lose momentum moving their orbits closer to the Sun or gain momentum increasing their orbital distance from the Sun. Changing orbits create collisions between planets. Moons form and collide with each other and with planets. Comets and asteroids form and smash into everything. But out of this chaos we get our current order.



Frost Line

There is an expected difference between how this works out for planets forming in the inner parts of a Solar System and how it works out for planets forming in the outer parts of a Solar System. In the inner parts, closer to the central star, it's hot. At these temperatures with no pressure, water molecules cannot take liquid or solid form. The solar wind from the forming star pushes these molecules along with helium and hydrogen molecules into the outer solar system. Planetesimals in this region wind up with little to no gas or water. That leaves metal and rock for these planets. In the outer parts, further from the central star, the water is frozen solid, and therefore behaves like rocks. Planets out here have metal, rocks and ice. This means that they're substantially more massive. This extra mass produces enough gravity to hold onto the gas as well. The dividing line is called the "Frost Line" or the "Snow Line", and its distance from the star is temperature dependent. It will be located where the temperature falls to around 250 degrees Kelvin (That's minus 10 degrees Fahrenheit). The hotter the star, the further out this line will be.

This explains the difference between the rocky planets in the Inner Solar System and the gas giant planets in the Outer Solar System. Our frost line is between 4 and 5 AU. That puts it at the far rim of the asteroid belt.





Meteorite Uranium and Lead


Meteorites are asteroid or comet material that have fallen to Earth. There are over 40,000 meteorites that we know about. Some have no Uranium. They can be used to measure the Solar System's initial lead ratios. The Canyon Diablo meteorite fits in this category. Some contain intact material from the circumstellar disk during the planetesimal building process. These are the ones that never went through a melt and rehardening process like all the rocks on the Earth and the Moon. The meteorite Allende fits into this category.

We'll start with Canyon Diablo - the meteorite that was responsible for Meteor Crater in Arizona. It is estimated to have fallen to Earth around 50,000 years ago. Fragments of the meteorite have been actively collected since the mid-1800s.



The largest fragment is the Holsinger Meteorite with a mass of 639 kg or 1400 lb. This iron meteorite contains troilite – an iron-sulfide (FeS) mineral that has almost no uranium. Since the mineral contains no uranium, all of the Pb present in the troilite is the Pb originally present when the meteorite formed. This includes the radiogenic isotopes Pb206 and Pb207 that decayed from U before the meteorite formed, as well as the natural non-radiogenic Pb204. Thus, using mass spectrometry as always, this Canyon Diablo troilite gives us the primordial ratios of $^{206}\text{Pb}/^{204}\text{Pb}$ at 9.307 and $^{207}\text{Pb}/^{204}\text{Pb}$ at 10.294. In fact, these two numbers are generally used as the standard for our solar system's original Pb concentrations.

Canyon Diablo Troilite



Then

$$\left[\frac{^{207}\text{Pb}}{^{204}\text{Pb}} \right]_i = 10.294$$

$$\left[\frac{^{206}\text{Pb}}{^{204}\text{Pb}} \right]_i = 9.307$$

Let $^{206}\text{Pb}_i$ = number of ^{206}Pb atoms initially

$^{207}\text{Pb}_i$ = number of ^{207}Pb atoms initially

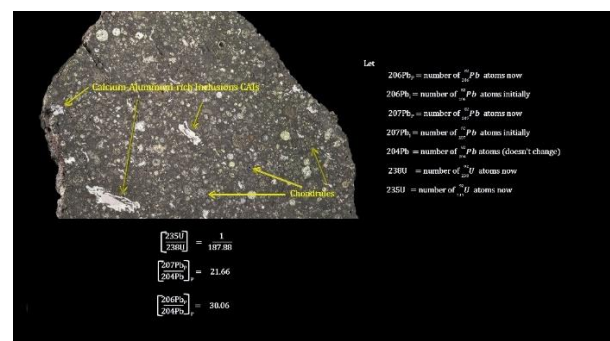
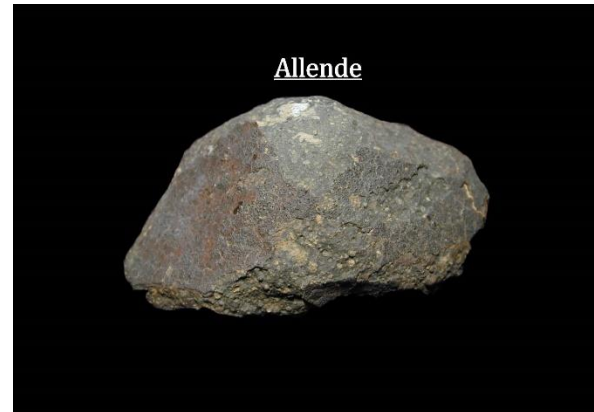
^{204}Pb = number of ^{204}Pb atoms (doesn't change)



In 1969, Allende created a fireball over the Northern Mexico sky. As the meteorite burst, numerous fragments rained down around the small village of Pueblito de Allende. Over 2000 kilos of debris have been found and new pieces are still being discovered every now and then. It's a type of meteorite called carbonaceous chondrite. That's a stony meteorite with lots of carbon and containing small mineral granules called chondrules. A lot of meteorites have experienced significant heat that melted and reorganized their minerals, but Allende was not one of them. Its pieces remained as they were when they formed.

Here's a slice of it. We're particularly interested in the little pale whitish-gray bits called calcium aluminum inclusions (CAIs). These are thought to be the very first solids that condensed in the circumstellar disk. Dating these would give us a starting date for the Solar System. Other interesting pieces of the meteorite are these round darker gray bits. These were the first liquid droplets to condensed out of the disk gas. They are also some of the oldest materials that formed in the solar system, but not as old as CAIs.

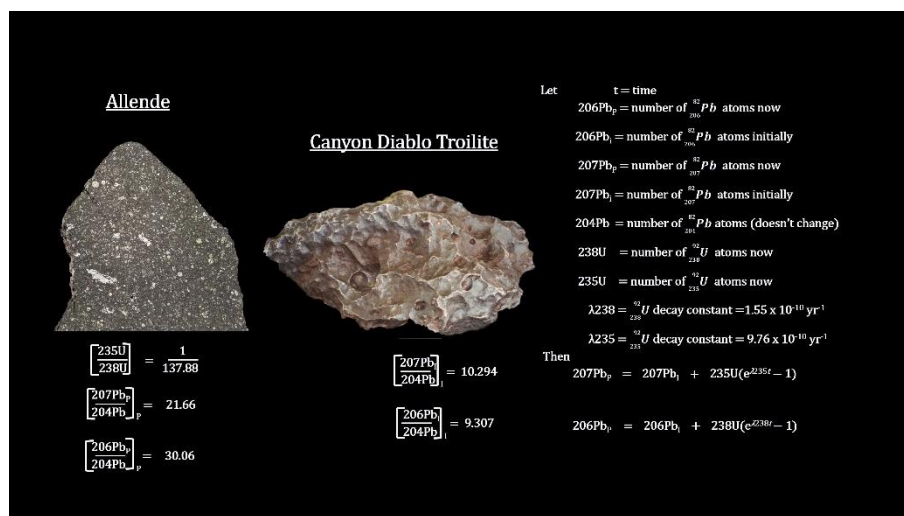
We are interested in two finding associated with Allende. One is the ratio of U238 to U235 isotopes. In Allende, it was around 137.88 U238s for each U235. This ratio has held up across Earth, moon and meteorite rocks. The other is the ratios of radiogenic Pb207 and Pb206 to natural non-radiogenic Pb204. A large number of these ratios were determined from the various CAIs and Chondrules, but we'll use just one: with Pb207 over Pb204 at **22.76** and Pb206 over Pb204 at **30.06**.





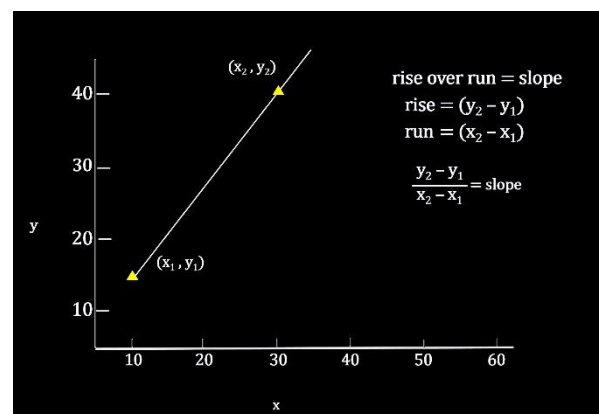
Meteorite Lead-Lead Dating

These ratios for current U and Pb from Allende and primordial Pb from Canyon Diablo are very important for dating meteorites. In the “How old is the Earth-Moon System”, we covered radioactivity, half-life, and the law of radioactive decay. It showed that the present amount of radiogenic lead in a sample will equal the initial amount plus however much gets created by the decay of the parent uranium. For Pb207, the parent is U235. For Pb206, the parent is U238. With this equation, we used Arthur Holmes’ system for Uranium decay into lead inside zircon crystals to date the oldest Earth and Moon rocks. But that won’t work for meteorites. They don’t have any zircon crystals.



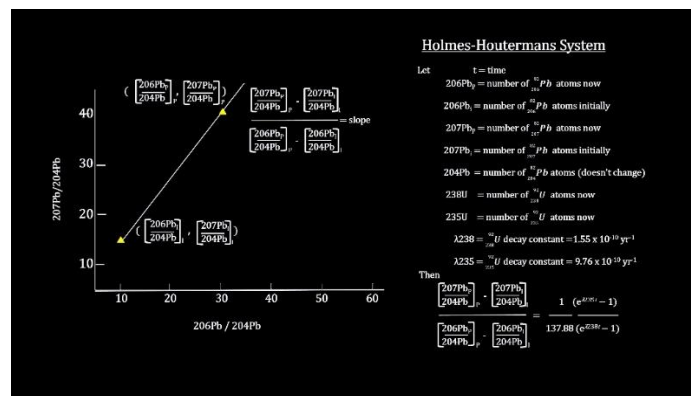
So, by the mid-1940s, Arthur Holmes and others had extended the U-Pb dating method into a Pb-Pb method called the Holmes-Houtermans System that took into account Pb204 the natural non-radiogenic lead isotope.

This is the system that tells us the age of meteorites, Earth, planets, and the entire solar system. So, we’ll take a minute to show how it works. The idea is to start with the uranium to lead growth equations and produce an equation that fits the definition for the slope of a line. [With values found in Canyon Diablo and Allende, we can measure the actual slope of this line and use it to find the age of the meteorite.]

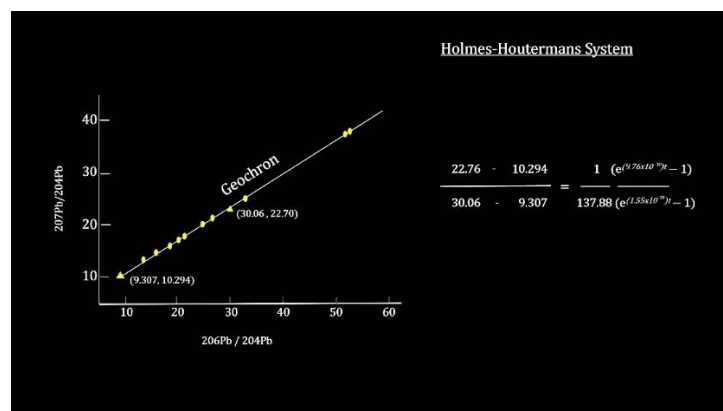




We start by dividing both sides of the growth equation by the number of Pb204 isotopes. We move the lead term on the right side of the equation to the left side - leaving only uranium on the right. We then divide the top equation by the bottom. This creates ratios on the left and right. We then preplace the U ratio on the right with the known value (1/137.88) found in Allende and elsewhere. This knocks out the need to measure U content all together. If we graph this equation using the 207 to 204 ratios as the y axis - and the 206 to 204 ratios as the x axis - we see that the left-hand side of our Holmes-Houtermans equation is the slope of a straight line.



The right-hand side depends only on time. For any given time t , we'll have a straight line. [That's why the line is called an isochron.] Replacing the initial lead ratios with the standard from Canyon Diablo we see that all lines will pass through this point. [And because U235 decays faster than U238, the Pb207 will accumulate faster than Pb206 and the slope will get steeper with time.] Now if we have t stand for the amount of time since the material formed, we can use the present-day lead ratios from an Allende CAI for the other point on the line. Other ratios from Allende also fall on this line. In fact, all materials formed around the time that this Allende CAI formed will fall on this line. That's why this isochron is called the Geochron. It represents the age of the solar system's planetesimals building blocks.





We now have an equation for t where t represents the age of the oldest material in the meteorite. This is a transcendental equation. It cannot be solved algebraically. But computer iteration processing gets us as close as we want. The solution gives us t equal to 4.567 billion years (plus or minus 70 million years)! This is the oldest age of all meteorites, meteors, asteroids, comets, moons and planets including the Earth, as well as the Sun. In fact, whenever you hear that the Sun is 4 and a half billion years old, that number came

from this Holmes-Houtermans process for radiometric rock dating.

Holmes-Houtermans System

$$t = 4.567 \times 10^9 \text{ years}$$

How Old is the Solar System

Now that we have a handle on star development from “How Old are Stars”, and some key dates from uranium/lead analysis of rocks and meteorites, we can now estimate the age of the solar system as a whole – give or take a few million years here and there. Our first data point is provided by uranium decay. A neutron star merger would have seeded our cloud with relatively equal amounts of U235 and U238. The time it takes for the ratio to reach today’s value of 1 U235 for every 137.88 U238 is 6 billion years.

Giant Molecular Cloud

Let

$$\frac{^{235}\text{U}_i}{^{238}\text{U}_i} = \frac{^{235}\text{U}_f}{^{238}\text{U}_f} = 1$$

Then

$$\frac{^{235}\text{U}_f}{^{238}\text{U}_f} = \frac{^{235}\text{U}_i}{^{238}\text{U}_i} e^{-(\lambda_{235} - \lambda_{238})t}$$

$$1/137.88 = e^{-(\lambda_{235} - \lambda_{238})t}$$

$$t = (\ln(1/137.88)) / (\lambda_{235} - \lambda_{238})$$

$$t = 6.00 \times 10^9 \text{ yr}$$

6.00

Time in billions of years

Let

$t = \text{time}$

$^{238}\text{U}_i = \text{number of } ^{238}\text{U} \text{ atoms initially}$

$^{235}\text{U}_i = \text{number of } ^{235}\text{U} \text{ atoms initially}$

$^{238}\text{U}_f = \text{number of } ^{238}\text{U} \text{ atoms now}$

$^{235}\text{U}_f = \text{number of } ^{235}\text{U} \text{ atoms now}$

$\lambda_{238} = ^{238}\text{U} \text{ decay constant} = 1.55 \times 10^{-10} \text{ yr}^{-1}$

$\lambda_{235} = ^{235}\text{U} \text{ decay constant} = 9.76 \times 10^{-10} \text{ yr}^{-1}$

Then

$^{235}\text{U}_f = ^{235}\text{U}_i e^{-\lambda_{235}t}$

$^{238}\text{U}_f = ^{238}\text{U}_i e^{-\lambda_{238}t}$

$\frac{^{235}\text{U}_f}{^{238}\text{U}_f} = \frac{^{235}\text{U}_i}{^{238}\text{U}_i} e^{-(\lambda_{235} - \lambda_{238})t}$

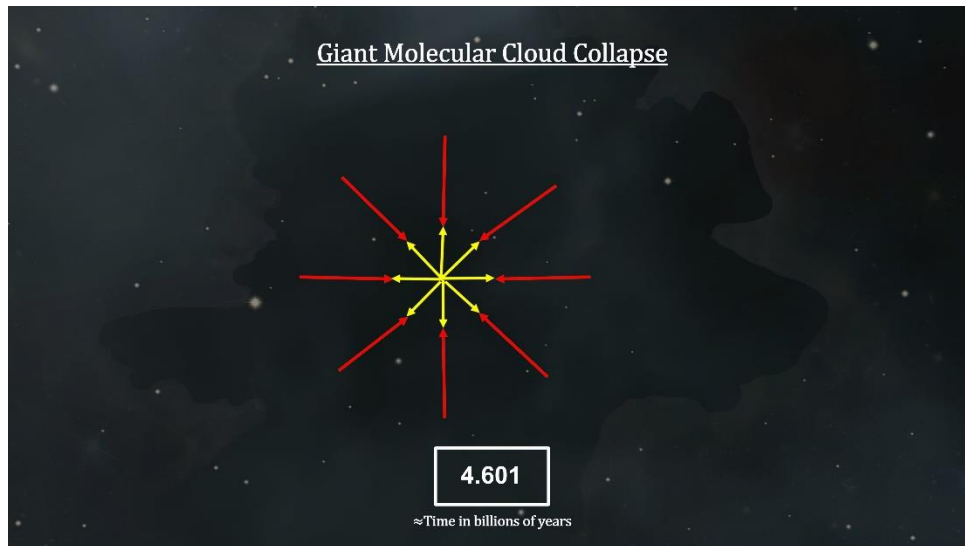
$1/137.88 = e^{-(\lambda_{235} - \lambda_{238})t}$

$t = (\ln(1/137.88)) / (\lambda_{235} - \lambda_{238})$

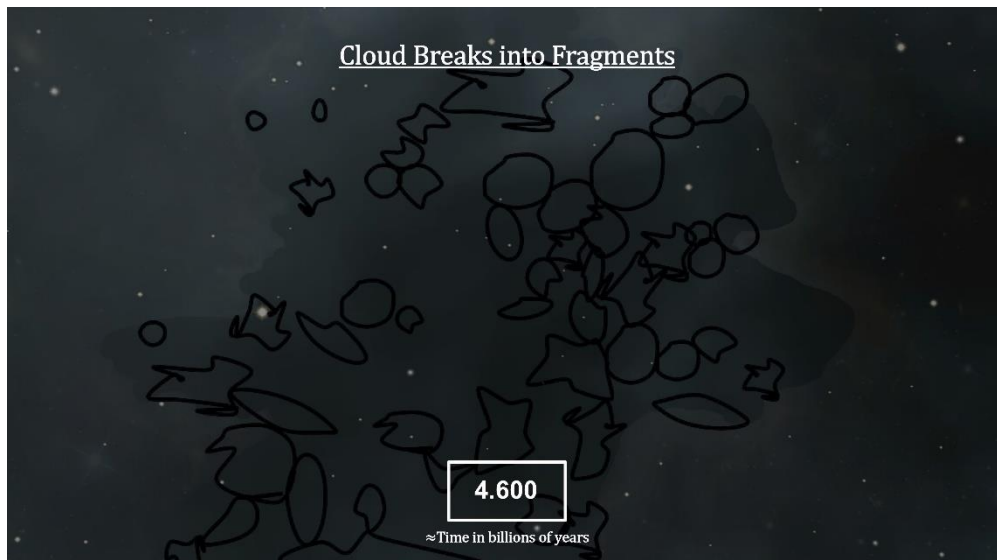
$t = 6.00 \times 10^9 \text{ yr}$



For almost 1.4 billion years, the cloud orbited the Milky Way in hydrostatic equilibrium. And then, for some as yet unknown reason, the equilibrium was broken and it started to collapse.

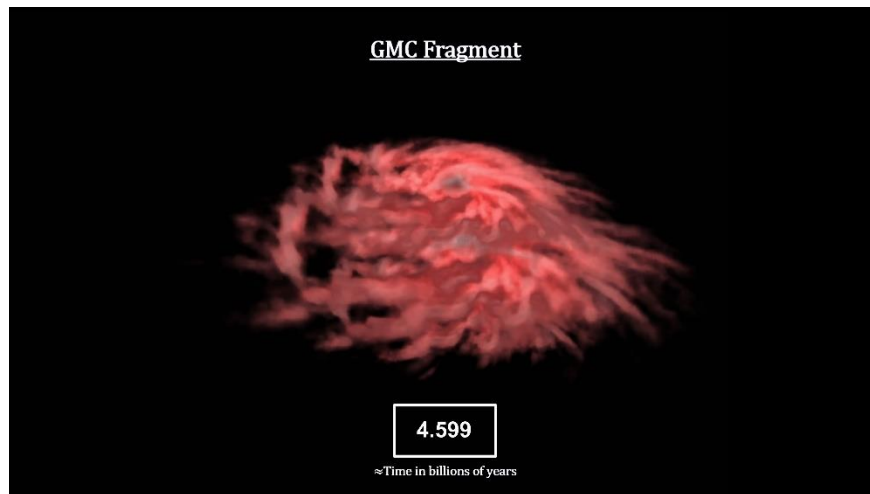


Within a million years, the collapsing giant molecular cloud broke up into fragments with our fragment being one of them.





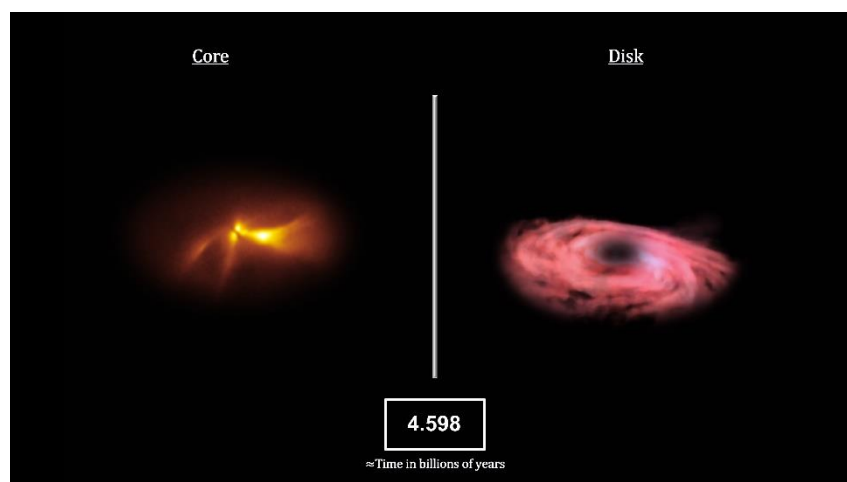
Over the next million years, a circumstellar disk formed around a central object accreting mass from the disk material orbiting around it.



From here on, we'll cover the development of the core object and the circumstellar disk in parallel.

Over the course of the next million years, the central object continued to accumulate matter and its core temperature reached 10 thousand degrees Kelvin. At this temperature, it began to shine via normal (non-nuclear) means. That made it a 'protostar'. It may have looked like this one just 950 light years away.

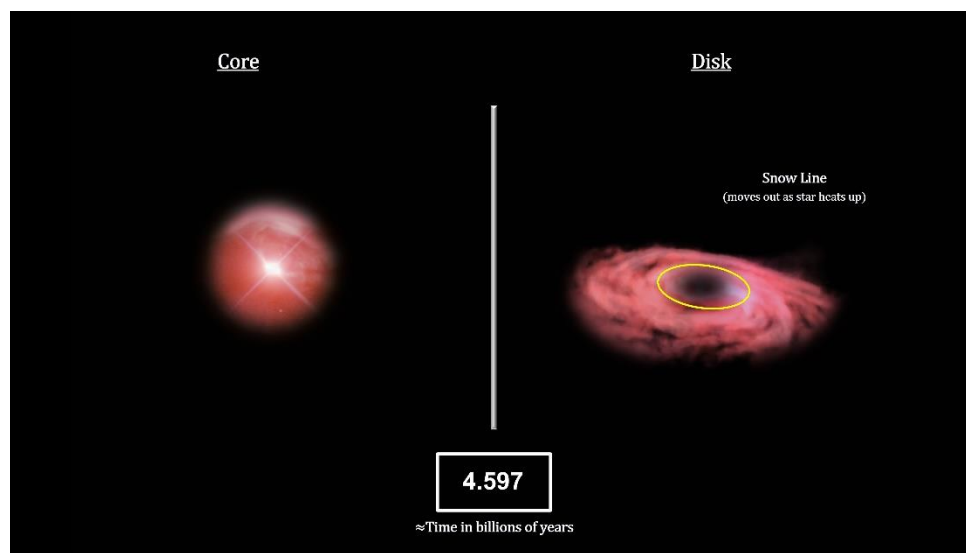
During this time, most of the matter continues to reside in the circumstellar disk. It's losing large quantities of material to the central object, but by the time the protostar forms, the disk still has 99% of the solar system's mass. Some dust may have been colliding and sticking together, but the vast majority of whatever formed in the disk during this period was eventually lost to the forming star.





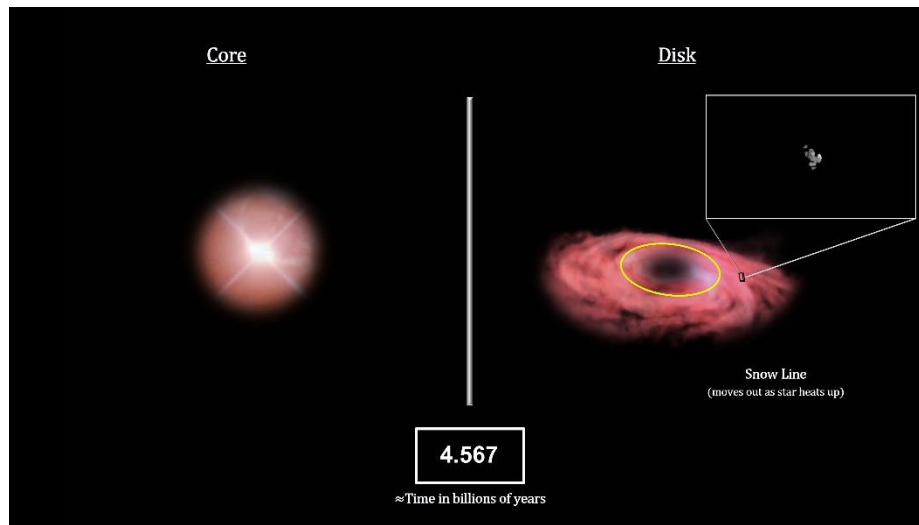
The Protostar phase does not last long for stars the size of our sun. Over the next million years, it accumulated massive amounts of matter from the disk and shrank significantly as gravity took hold. Its core temperature rose to 5 million K. This put the sun into its T-Tauri phase named after the star T-Tauri. In fact, it may have looked like T-Tauri. The young sun was still growing by accreting large amount of material from its surroundings, so it was not yet stable. Unlike the short lifespan for protostars, T-Tauri stars can last for a hundred million years.

During this phase, the disk experienced a growing solar wind from the developing star. This wind started pushing on the lighter gas and dust close to it, forming a snow-line beyond which water ice could form. With the sun at only 5 million K, this line would be much closer to the star than it is today.



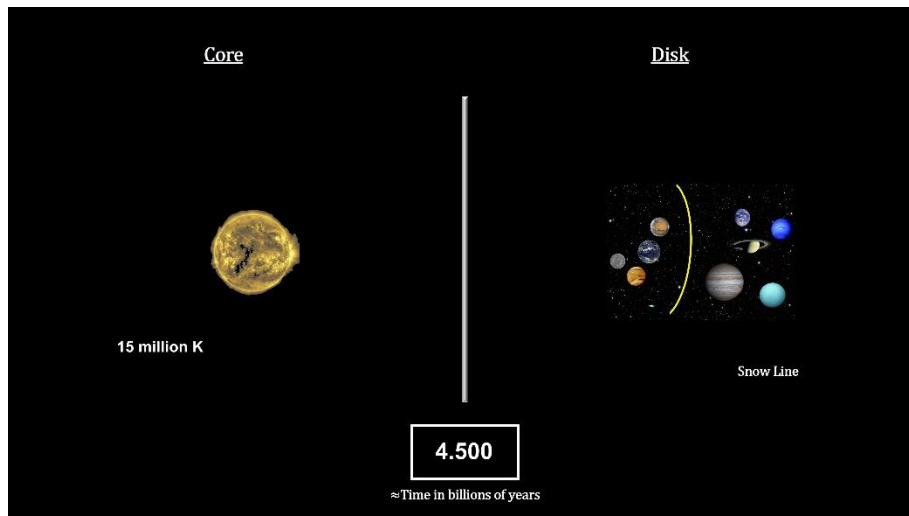
Over the next 30 million years, the Sun's core temperature would have reached 10 million K.

Throughout the disk, some dust grains began to stick to each other, forming larger particles. These particles continued to randomly collide and stick creating planetesimals reaching the size of boulders or small asteroids. The oldest of the starting material found so far was in the Allende meteorite and dated via their lead isotope contents to be 4.567 billion years old. By convention, astronomers use this date for the age of the sun and its solar system (often rounded up to 4.6).



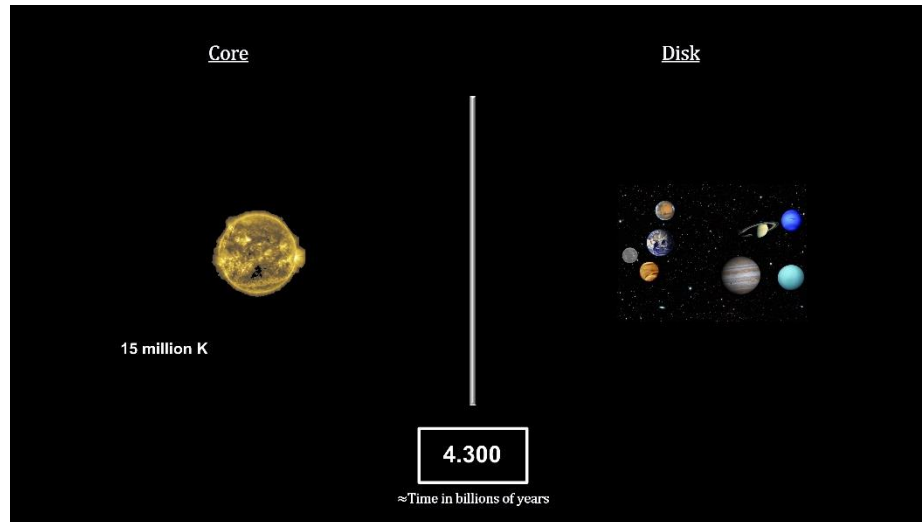
Estimates are that the sun remained in its T-Tauri phase for an additional 67 million years as it migrated to the Main Sequence. In that time, it reached 15 million K at its core. As the Sun's solar wind picked up, it dispersed the remaining gas and dust around it back into the interstellar medium. This ended mass accumulation, and the Sun settle into hydrostatic equilibrium.

During this period in the disk, the forming of planetesimals increased as the objects began to attract each other via gravity. Objects grew to planet sizes and sweep out the debris in the vicinity of their orbits. The disk experienced a chaotic period of collisions that resulted in 9 major planet sized objects along with dozens of moons, and millions of asteroids and comets. In addition, as the Sun heated up, the snow-line moved out to where we find it today just outside the asteroid belt. This line separated the 4 waterless inner solar system planets from the 5 water rich outer solar system planets.

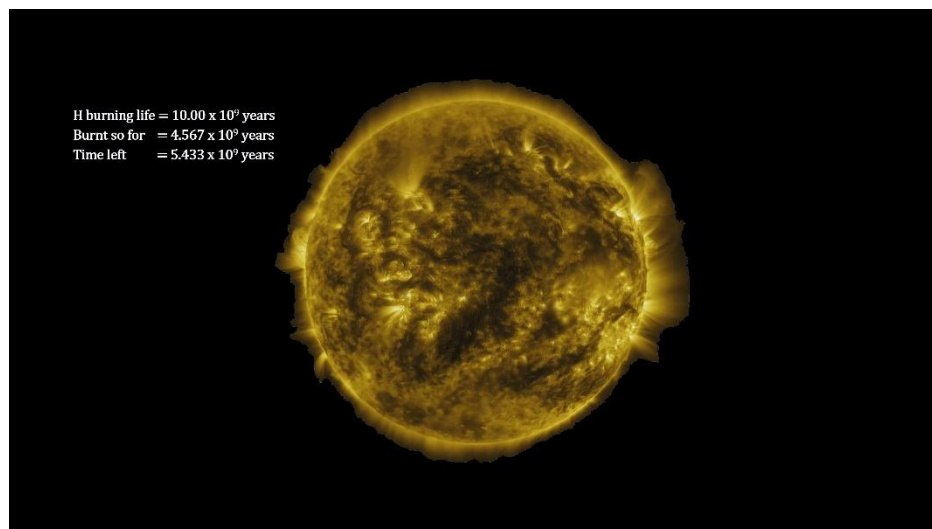




The Giant Impact Hypothesis has a collision between the Earth and a Mars sized planet liquifying the crust of both planets and forming the moon from ejected matter. Based on uranium-lead dating of zircon crystals found in Australia and on the moon, this happened 4.3 billion years ago. That would be 200 million years after the original Earth formation.



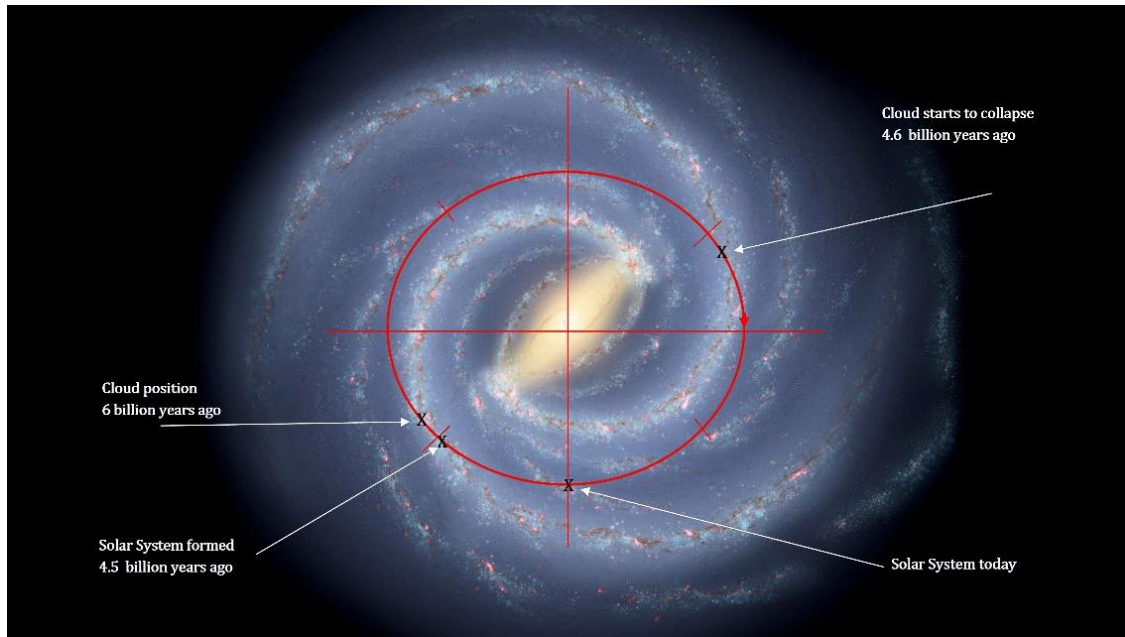
Given the mass of our Sun, we know that, in the beginning, it had enough hydrogen to shine for a total of 10 billion years. We now figure that it has been burning for 4.6 billion years. Therefore, we can expect that will burn for 5.4 billion more years before it runs out of fuel.



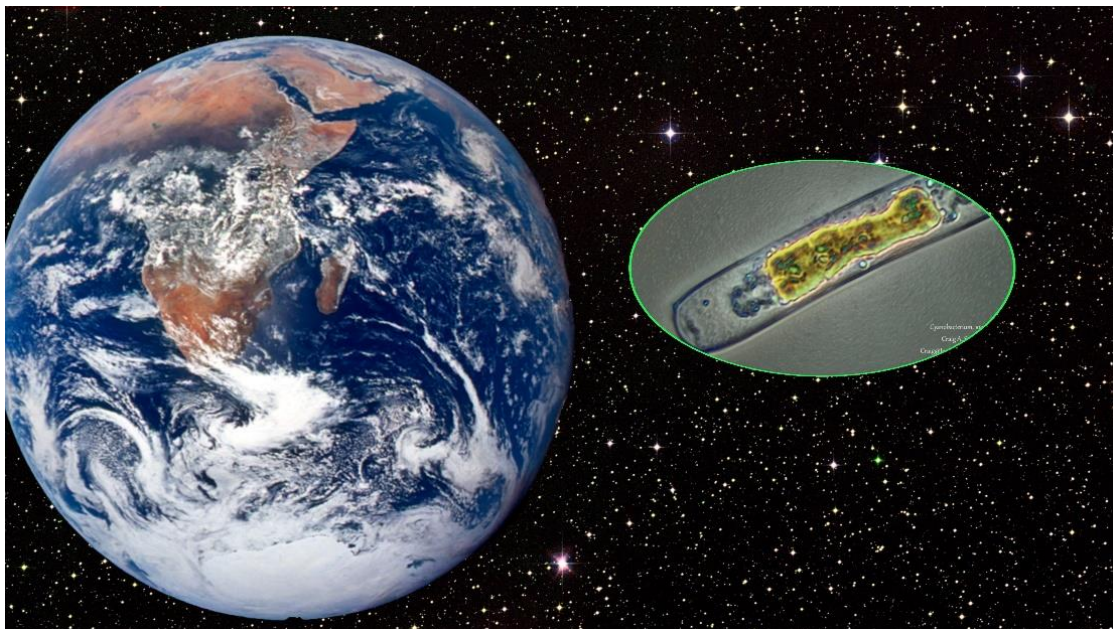
Here's our position in the Milky Way today. I am impressed by how we have been able to reconstruct our solar system's formation. We started with a Giant Molecular Cloud rotating around the Milky Way every 213 million years 26,000 light years from the center. It was seeded with



uranium around here (a little over an eighth of a revolution from our current location). It rotated an additional six and a half times before our cloud segment started to collapse here where we find the Perseus arm today. It took only half of a revolution more to form the entire Solar System.



Today, we have a beautiful planet with a Sun that will sustain us for billions of years. But even more spectacular than the rise of the Earth over a 300 hundred million year period, is what happened in the 4.3 billion years since our temperate, watery Earth formed. Life!





Music

@00:00 Beethoven: Symphony No. 6 in F Major, Op. 68 Pastoral I. Allegro ma non troppo; London Symphony Orchestra; Josef Krips, Conductor; from the album Ludwig van Beethoven: Symphony No. 6 (Pastorale), Egmont Overture - Incidental Music, Op. 84, Fidelio Overture/March, Op. 72, Coriolan Overture, Op. 62, 2009

@12:47 Puccini, Giacomo: Edgar Act 1 – Prelude; Radio-Symphonie-Orchester Berlin and Riccardo Chailly; from the album Puccini Without Words 2006

@23:45 Schubert: Symphony No. 5, Andante; from the album “Meditation: Classical Relaxation” 2010

Greek letters:

- α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ σ τ υ φ χ ψ ω

- Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ Υ Φ Χ Ψ Ω

⇒ → ± ⊙ ∞ ↗ ∃ ∄ ∈ ∉ ∫ ∫ ∫ ≅ ≥ ≤ ≈ ≠ ≡ √ ∛ ∼ ∝ ℏ ÷

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