General Relativity I - Geometry

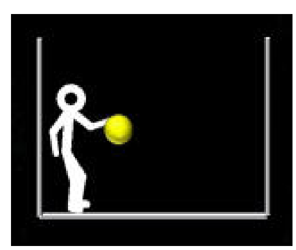
{**Abstract:** *In this segment of the "How Fast Is It" video book, we cover the geometry of general relativity. We start with the Elevator Thought Experiment, and show how it represents a gravitational field and how it predicts the bending of light. This sets the stage for the Equivalence Principle. This leads to the reconciliation of Newton's two definitions for mass. Which, in turn, leads to the idea that the existence of a mass bends space. To understand the bending of space, we cover the basics of Euclidian and non-Euclidian Riemann geometry. We include spherical and hyperbolic geometries along with the nature of their respective geodesics. We actually measure geodesic deviation above the Earth. For a fuller understanding, we cover the definition of metrics and curvature in terms of tensors. With the general Riemannian Curvature Tensor in hand, we find the subsets that reflect the behavior of space within a volume. We then cover how Einstein mapped this geometry to space-time to produce the Einstein Curvature Tensor. And finally, we describe the Energy-Momentum tensor that identifies the nature of a volume of matter-energy, which is the source of the space-time curvature. Setting these equal to each other with an appropriate conversion factor gives us Einstein's general relativity field equations.* }

Introduction

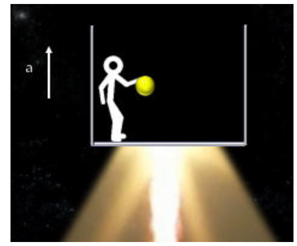
*[**Music**: Stravinsky - Apollon Musagete, Scene 2 Apotheose - Apollo, Leader of the Muses) is a ballet composed between 1927 and 1928.]*

We're zooming into the center of the Sculptor void. It's the largest void in the nearby Universe. This will take us to a place with as little gravity as we can find.

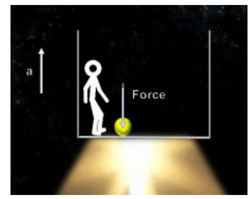
Now imagine that you're in an elevator at the center of this great void. You're not accelerating. You're weightless in an inertial frame – stuck to the bottom with Velcro boots. You're holding a ball. If you let go, it would float in place.

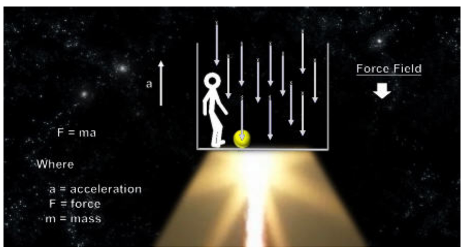


Now imagine that the elevator starts accelerating at a constant velocity. You will now feel a force pulling you down.



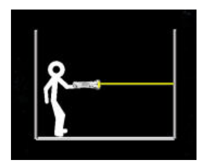
Let go of the ball again. This time it falls to your feet. No matter where you let go of the ball, it will feel the same force and fall to the bottom of the elevator.



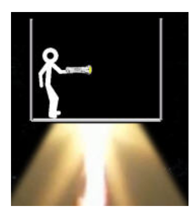


In other words, at every point in space inside the elevator, a force is felt. This is a force field. The elevator's acceleration has created a force field inside the elevator. The acceleration of the ball follows Newton's 2nd law of motion - Force = mass x acceleration.

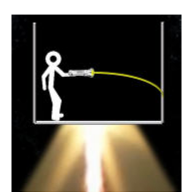
Now turn off the engines to return to an inertial frame. You're not accelerating. It's dark. You turn on a flashlight and watch the light reflect off the far elevator wall.

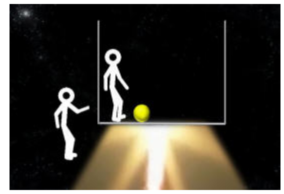


Start the elevator accelerating again at a constant rate.

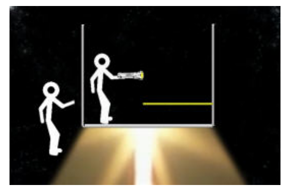


Turn on your flashlight and watch the light beam hit the far wall further down than it did when you were at rest. The light bends down!

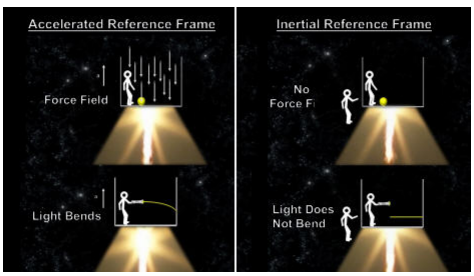




Now let's take a look at what a person outside the elevator looking in would see when we drop a ball and shine a light. The person outside the elevator is not accelerating. When the person in the accelerating elevator lets go of the ball, the person outside the elevator sees that it sill hovers in place, just as before. He sees that the elevator moved up to hit the ball. There is no force acting on the ball. There is no force field inside the elevator.

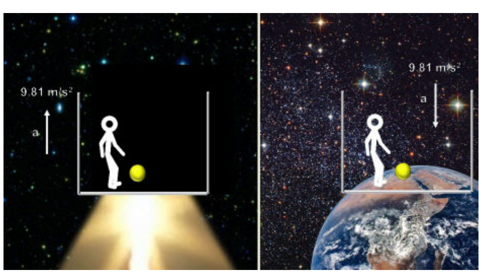


And when the person in the elevator turns on the flashlight, the person outside the elevator sees the light travel in a straight line as before. He sees that it's the elevator's wall moving up that causes the light to hit it at a lower point. The light does not bend.



Who is right? Before Einstein's GTR, we would have said the inertial observer was correct and the person in the elevator was fooled into thinking he is in a gravitational field. But according to GR, they are both right in their own reference frame. Gravitational forces have materialized for the person in the elevator due to its accelerated motion.

According to GR, this gravitational field is as real as one created by the existence of a massive object.

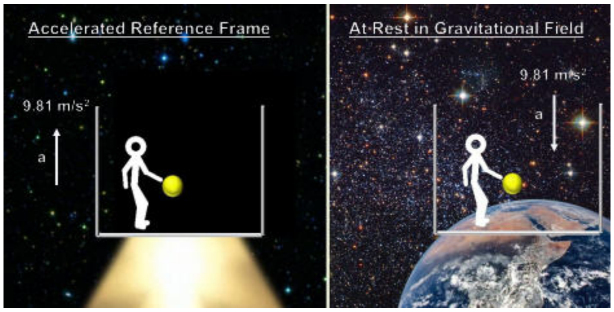


To see this, let's compare what the person in the elevator is experiencing and what a person at rest in a gravitational field would experience. We know that Earth's gravity near the surface accelerates objects at 9.8 m/s^2. If we set the acceleration of the elevator at 9.8 m/s^2, the occupant would experience the weight he feels on Earth, and the ball would fall at the same rate as it does on Earth.

Equivalence Principle

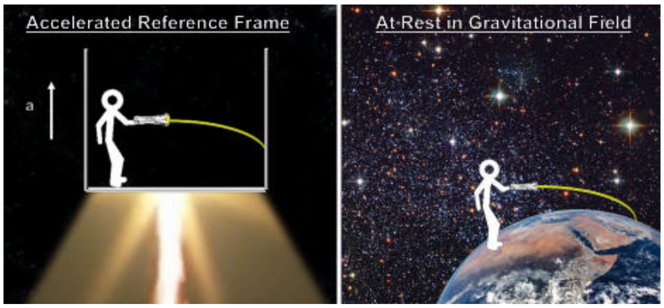
[Music: Dvorak - Songs My Mother Taught Me from Gypsy Melodie - Written in 1880.]

In fact, the person in the elevator cannot tell the difference between the two situations. Is he out in space being accelerated by some force or is he at rest on Earth being accelerated by Earth's gravity? As far as the laws of physics are concerned, being accelerated and sitting still in a uniform gravitational field are equivalent. This is Einstein's Equivalence Principle. It is a generalization of SR that holds that the laws of physics were the same for all inertial reference frames. With GR we hold that the laws of physics are the same for all reference frames no matter what their relative motion.



The equivalence principle has a number of implications. One of the most significant for us is that it tells us that because light bends in the elevator, it will bend in a matter generated gravitational field as well.

You'll recall from our segment on Special relativity, that light speed in a vacuum being a constant lead directly to time dilation, space contraction, unusual velocity addition, and the end of simultaneity. With GR, we'll see how the bending of light in a gravitational field has its own set of even more dramatic changes to our understanding of the physical world.

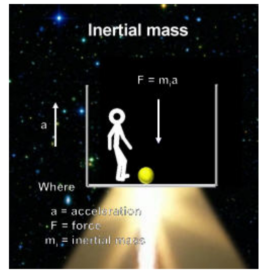


[Music: Giacomo Puccini- O mio Babbino caro ("Oh My Beloved Father") - a soprano aria from the opera Giann Schicchi (1918).]

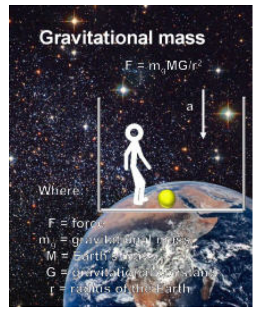
Mass

One of the motivations for the equivalence principle was the long standing problem with Isaac Newton's classical physics when it came to mass. Newton defined two kinds:

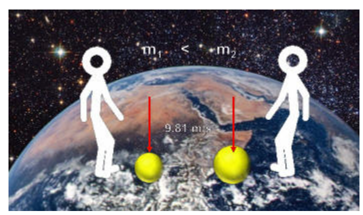
- Inertial mass was defined by how much force it took to accelerate an object. It is described by the Force equals mass times acceleration formula.



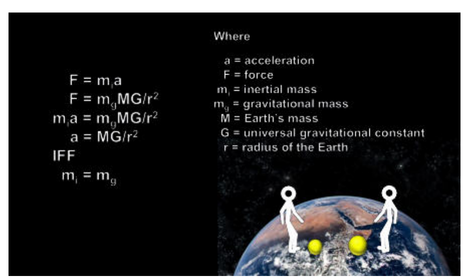
- Gravitational mass was defined by how strong an attractive force it exerted on other objects. It is described by Newton's universal gravitation formula. These are two very different definitions for mass.



But, ever since Galileo's experiments with falling objects, we have known that masses do accelerate at the same rate no matter how massive they are.

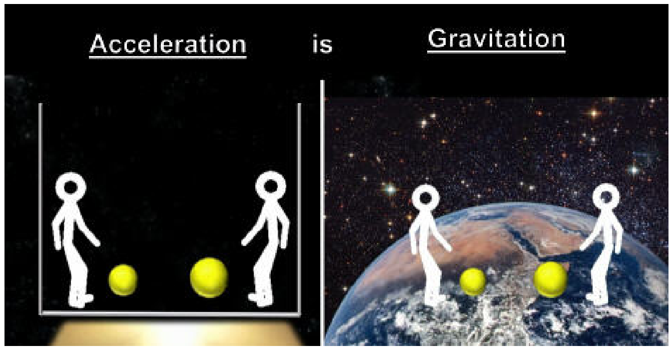


That objects with different inertial masses fall at the same speed in a gravitational field can only happen if the inertial mass and the gravitational mass for any object are equal. Measurements to this day show that these two kinds of mass are indeed equal. The data lead Newton to declare them equal, but he could not explain why they were equal.



Einstein felt that before you can declare two things equal, you need to demonstrate an equality in the real nature of the two concepts. In other words, we can only say they're equal after their real nature is found to be equal.

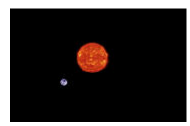
His equivalence principle does just that. Acceleration and gravitation are the same, and therefore the mass associated with acceleration and the mass associated with gravitation will naturally be the same as well. Problem solved. But a new set of non-intuitive consequences followed.



Early Considerations

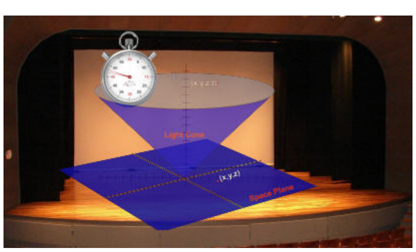
From antiquity into the eighteenth century, it was believed that the idea of empty space is a conceptual impossibility.

Space is nothing but an abstraction we use to compare different arrangements of the objects. Concerning time, it was believed that there can be no lapse of time without change occurring somewhere. Time is merely a measure of cycles of change within the world.



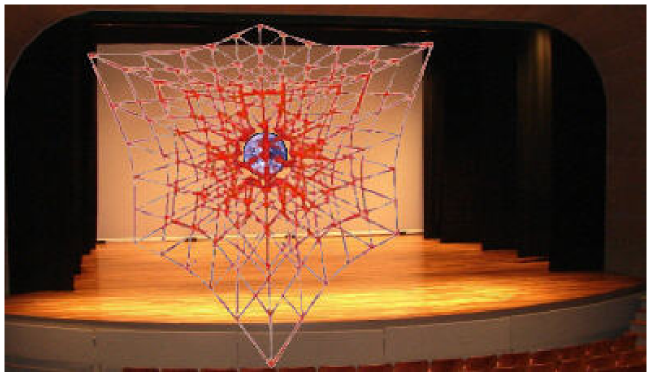
Then, in 1686, Isaac Newton founded classical mechanics on the view that *space* is real and distinct from objects and that *time* is real and passes uniformly without regard to whether anything moves in the world.

He spoke of *absolute space* and *absolute time* as a stage within which matter existed and moved as time flowed at a constant rate. It was understood that space and time tell matter how to move, but matter has no effect on space and time.



We have seen that SR broke the paradigm of absolute space and absolute time, because the constancy of the speed of light required a tradeoff between space and time across different inertial frames. In addition, and more to our point, the idea that space and time act on matter, but that matter does **not** act on space and time, troubled Einstein.

With these considerations in mind, and noting that light curved in a gravitational field, Einstein proposed that the mass of an object does indeed act on the space and time it exists in. Specifically, he proposed that the presence of matter curves space-time.

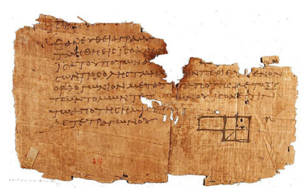


To see how this can be, we need to examine the geometry of space a little closer. We'll start with the Euclidean geometry of flat surfaces and generalize it to curved space-time.

*[**Music**: Debussy - Prelude to the Afternoon of a Faun - a symphonic poem. It was first performed in Paris in 1894.]*

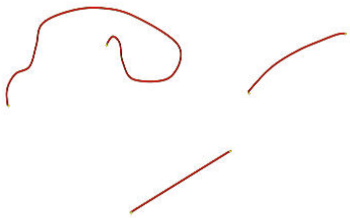
Euclidian Geometry

Euclid lived in the Greek city of Alexandria in Egypt around 2,300 years ago. He spent his life studying and teaching geometry. He published his ideas in a book called "Elements". To this day, it is the foundation for our understanding of geometry and mathematical processes in general.



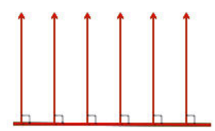
Oxyrhynchus papyrus showing fragment of Euclid's *Elements*, AD 75-125 (estimated)

In the early to mid-1800s, new geometries were studied by mathematicians like Johann Carl Friedrich Gauss (one of the greatest mathematicians of all time) and his colleague Georg Friedrich Bernhard Riemann. We'll cover a bit of what they found.



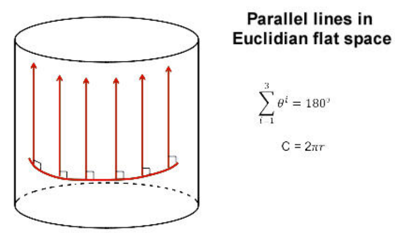
To understand the differences between Euclidian geometry and other possibilities we start with points and connect them with lines. The shortest distance between two points is the line with the least curves. In Euclidian geometry, this is a straight line. In this case, the least curvature is no curvature at all. Another name for the shortest line between two points is the geodesic.

If we draw geodesics that are each perpendicular to a third line, they will be parallel to each other. They will never cross, even if they are extended to infinity. This is a key characteristic of flat Euclidean space.



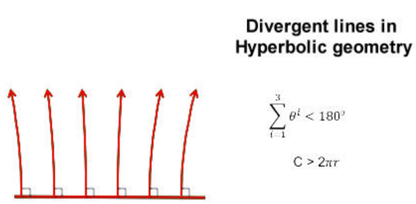
What we are talking about here is the intrinsic characteristics of the geometry. Things like the sum of the angles of a triangle is 180 degrees and the circumference of a circle is 2 π times its radius. We can bend this 2 dimensional surface into a third dimension and give it the look of curved space.

But this curvature is extrinsic by nature. The intrinsic geometry is still flat: parallel lines remain parallel; the sum of the angles of a triangle is still 180 degrees; and the circumference of a circle is still 2 π times its radius.

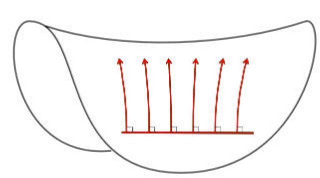


But there are other possibilities. One possibility for a different geometry supposes that the parallel geodesic lines are diverging - getting further apart.

It's as if space was being stretched between the lines. And the further up the lines you go, the more the space is stretched. Here the sum of the angles of a triangle is less than 180 degrees and the circumference of a circle is more than 2 π times its radius.

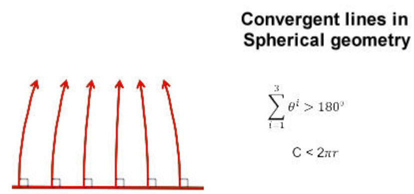


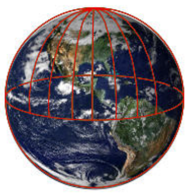
This is hyperbolic geometry. It represents space with a negative curvature. The best example of this is the surface of a saddle or a potato chip.



Another possibility for a different geometry supposes that the parallel geodesic lines are converging and will eventually meet.

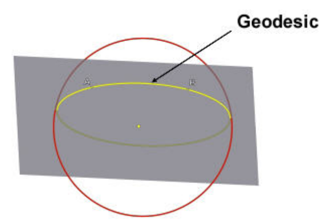
It's as if space was being compressed between the lines. And the further up the lines you go, the more the space is compressed. Here the sum of the angles of a triangle is greater than 180 degrees and the circumference of a circle is less than 2 π times its radius. This is spherical geometry. It represents space with a positive curvature.



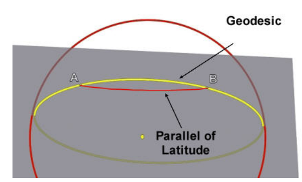


The best example of this is the surface of a sphere like the Earth itself. Here the curvature is constant throughout the surface. The base line is the equator. The perpendicular lines are the lines of longitude, and they meet at the North Pole.

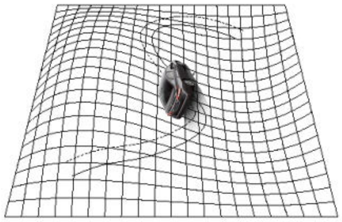
Here's the best way to find the geodesic between two points on a sphere. First intersect the sphere with a plan that contains the two points plus the center of the sphere. The intersection is called a Great Circle. The segment of the circle that connects the two points is the shortest distance between the two points.



A parallel of latitude line between the two points would be longer.



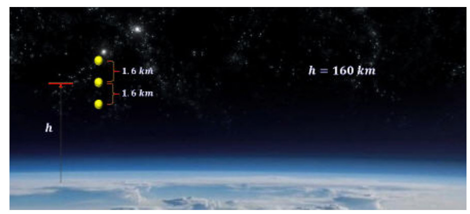
This is why planes in the northern hemisphere travel north and then back south to get to a destination at the same latitude rather than travel due east or west. They have chosen the shortest distance between the two points to save on both time and fuel. Some geodesic routs can save up to a thousand kilometers.



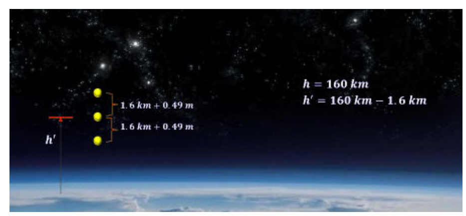
Of course, when it comes to a more generalized curved space, the curvature changes from place to place. If we were to put a car on a curved surface like this one, and lock its steering wheel to go straight, it would naturally follow the space's geodesic line from its starting point. When it's done, it will have traveled the shortest distance between its starting point and its ending point.

Measuring Geodesic Deviation

Above the Earth we can measure geodesics and their deviation using test particles. For example, if we place three test particles vertically above the Earth's atmosphere, and separate them by a small amount, we can see what happens when they all fall freely along their geodesic lines towards the center of the Earth.



Because the particles closer to the Earth will feel a slightly stronger gravitational attraction than the particles further up, they will accelerate faster. The distance between them will increase. This shows that the curvature is negative along this line in space above us.

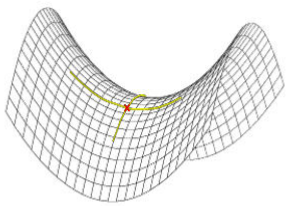


If we place the three test particles horizontally along an east-west line with the same starting separation, we can see what happens when they all fall freely along their geodesic lines towards the center of the Earth.

Because the particles are moving to the same point, the distance between them will decrease. This shows that the curvature is positive along this line in space above us. The same would be true for the same reason if we started with a horizontal north-south line.

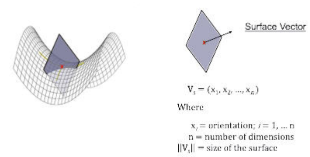
Interestingly, if we sum the three curvatures for the three special dimensions, we get a total curvature of zero!

Tensors

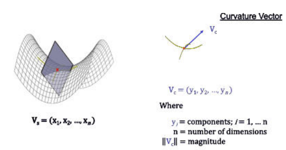


We see that at any point in n dimensional space, there are n independent directions, and each line through a point can have a different curvature. Here we picture just two. In one direction we have a positive curvature. In another it is negative.

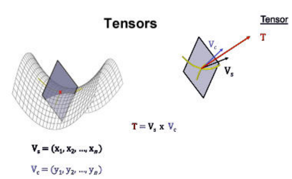
We can also construct a surface for each dimension. Each surface has a size and an orientation. We use vectors to mathematically represent the size and orientation of surfaces.



And for each surface, we can construct a vector that represents the curvature of the lines through a point on the surface.



Multiplying these two of vectors creates a mathematical object called a tensor.



$$T = V_s \text{ x } V_c = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}$$

Where $\quad g_{ij} = x_i y_j$

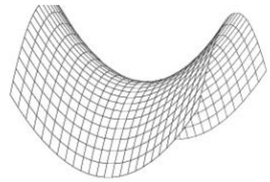
The power of tensors lies in two basic characteristics: first, they carry a great deal of information. And second they are invariant when the coordinate systems are changed. In other words, they remain constant across all kinds of changes in how we're looking at any particular situation.

Riemannian geometry

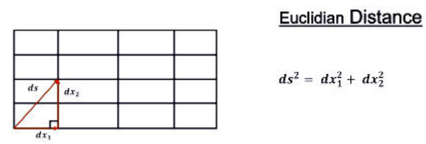
Riemann developed the mathematics for this generalized space with any number of dimensions. Basically it is a three step process.

- First we define a metric for the space that allows us to measure distances.

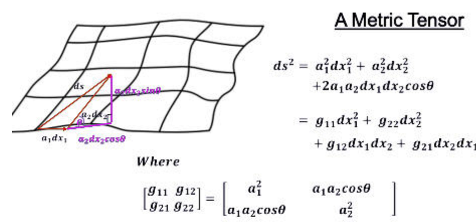
- Second, we use the distance metric to find the geodesics for the space.
- And third, we use the geodesics to define what we mean by curvature.



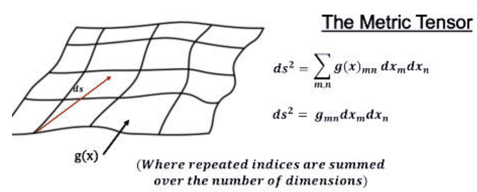
These spaces are smooth. By that I mean there are no abrupt changes. Given that, we can always zoom into a curved space to the point that the small piece we are looking at is flat. And in flat space, distance between any two points is defined by the Pythagorean Theorem.



We then generalize by adding coefficients to take into consideration the different scales for lines in different directions and the fact that the lines no longer cross to form right angles. This is called a metric tensor.



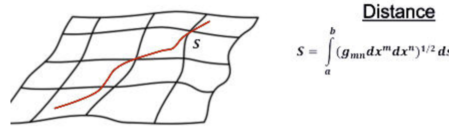
And finally, we extend the number of dimensions and generalize to coefficients to be functions of location to take into account curved and changing coordinate systems. This generalized metric tensor is the foundation for non-Euclidean geometry, its geodesics, and its curvature at any point.



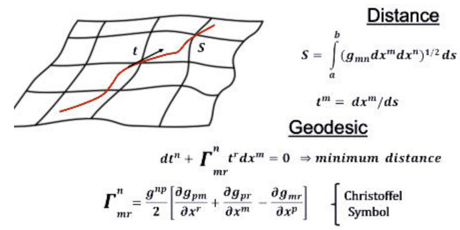
[Music: Robert Schumann - Traumerai (Dreaming) - Written in 1838, it was featured in the 1947movie "A Song of Love".]

Geodesics

With the metric tensor, we can measure distances between any two points by adding up all the small distances along the way.



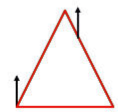
Taking the formula and finding its minimum is an exercise in calculus that gives us the shortest distances. These are the geodesic lines.



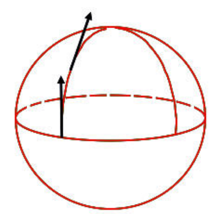
Curvature

Now that we have a way to measure distance and find geodesics, we can determine a space's curvature. Riemann used a concept called parallel vector transport.

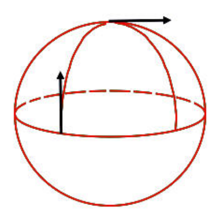
Picture moving a vector around a triangle in flat space in such a way that it remains parallel to the starting vector. By the time we get back to the start, we have the exact same vector as we started with.



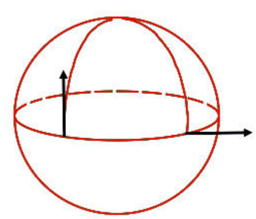
Now repeat this same exercise on a curved surface like the surface of the earth. Start at the equator and point the vector in front of you facing north.



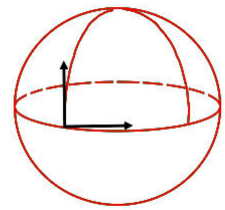
Move north along the geodesic longitude line. When you reach the North Pole, turn 90 degrees to your right. To keep the vector pointing in the same direction, it is now pointing to your left 90 degrees.



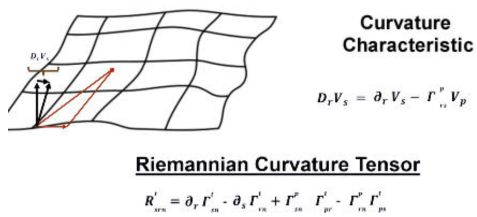
Move south along the geodesic longitude line to the equator and turn right 90 degrees again. The vector must now be made to point to your rear in order to keep it parallel to the original direction.



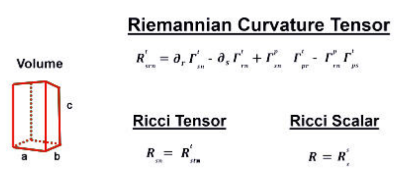
Now walk west along the equatorial geodesic until you reach your starting point. Clearly, the vector is no longer the same as when you started. The difference is the measure of the curvature of the space you travelled.



Riemann developed the tensor that precisely measures how much the components of a vector change when it is parallel transported along a small closed curve. This is called the Riemann curvature tensor.

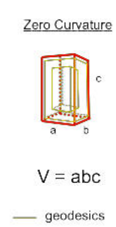


A subset of this tensor was developed by a mathematician named Gregorio Ricci-Curbastro called the Ricci tensor that compares the volume of space for a given Riemannian curvature to the volume of space in Euclidian geometry. Where Riemann gives us the curvature for every geodesic, Ricci gives us an average for a volume. Averaging this, we get a volume scalar. With this we can calculate the amount by which the volume deviates from what it would be in Euclidean space.



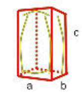
*[**Music**: Ron Grainer - The Doctor Who Themes - Created in 1963, it was one of the first electronic music signature tunes for television.]*

- For example, in Euclidian flat space, a cuboid's volume is a times b times c. The yellow lines represent geodesics inside the box.



- The volume is less than this, if the Ricci curvature of the interior region is positive. In other words, it's smaller on the inside.

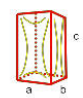
Positive Curvature



V < abc

- The volume is more than this, if the Ricci curvature of the interior region is negative. In other words, it's larger on the inside.

Negative Curvature



V > abc

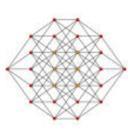
GR Field Equations

*[**Music**: John Williams - Across the Stars (Love Theme from Star Wars Episode II).]*

The Einstein Tensor

The Equivalence Principle led Einstein to the position that the presence of matter curved space-time, and that a body - free from all forces - travels geodesics in this curved space. With Riemannian geometry and the Ricci tensor, he had what he needed to develop the curvature side of the equation.

First, we fix the generalized n-dimensional coordinates to our four space-time coordinates. The convention is for numbers to run from 0 to 3, with 0 being the index for the time component.



n dimensional space

Riemannian Curvature Tensor

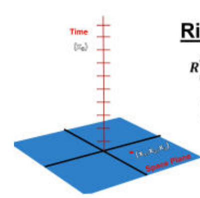
$$R^r_{srn} = \partial_r \Gamma^r_{sn} - \partial_s \Gamma^r_{rn} + \Gamma^p_{sn}\Gamma^r_{pr} - \Gamma^p_{rn}\Gamma^r_{ps}$$

Ricci Tensor

$$R_{sn} = R^r_{srn}$$

Ricci Scalar

$$R = R^s_s$$



4 dimensional space-time
$(x_0, x_1, x_2, x_3) = (t, x, y, z)$

Riemannian Curvature Tensor

$$R^\sigma_{\mu\nu\tau} = \partial_\nu \Gamma^\sigma_{\mu\tau} - \partial_\mu \Gamma^\sigma_{\nu\tau} + \Gamma^\lambda_{\mu\tau}\Gamma^\sigma_{\lambda\nu} - \Gamma^\lambda_{\nu\tau}\Gamma^\sigma_{\lambda\mu}$$

Ricci Tensor

$$R_{\mu\nu} = R^\tau_{\mu\nu\tau}$$

Ricci Scalar

$$R = R^\nu_\nu$$

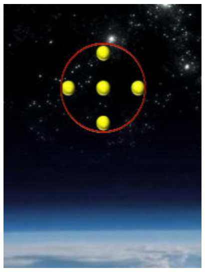
Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}$$

Then we find the volume of the 3 spatial dimensions for a given time dimension. This is the Einstein tensor.

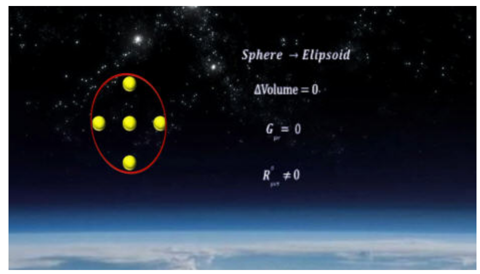
Einstein Tensor

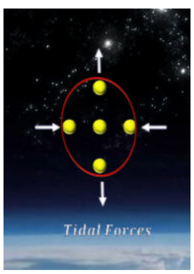
$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}$$



Physically, this tensor determines how the volume of a small group of particles in free fall along geodesics will change. [It is built from the Riemannian curvature tensor that quantifies curvature, but it is not actually a measure of curvature itself.] Our little seven particle experiment helps illustrate this important fact.

Here we see the change in the volume as the particles diverge and converge along various geodesics. This changes the shape of the volume from a sphere to an ellipsoid but the total volume remained unchanged. The Einstein tensor is zero even as the curvature of the space is not zero.

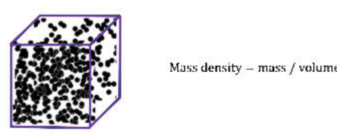




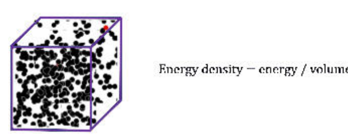
By the way, this sphere to ellipsoid phenomenon is called a tidal effect, because it is how our moon creates tides on the Earth.

Energy-Momentum Tensor

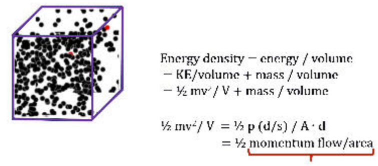
In Newtonian physics, the force of gravity was created by mass, or more precisely, mass density – the amount of mass per unit volume.



In GR, we need to change this from mass density to energy density, because of the equivalence between mass and energy and to take into account the motion or kinetic energy of the masses.

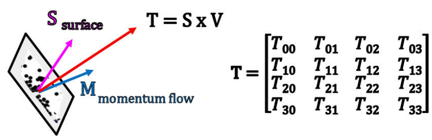


So in addition to calculating the mass-energy density of a volume of space, we need to account for the flow of energy through each surface of the volume.



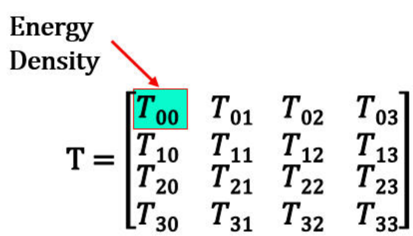
This information can be packed into a 4x4 matrix known as the Energy-Momentum Tensor [or Stress-Energy Tensor]. Each element represents the flow of momentum across a surface.

Energy-Momentum Tensor

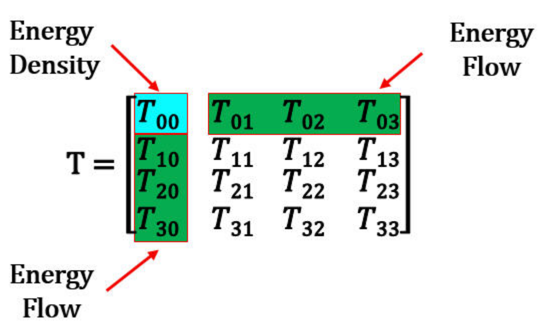


$T_{\mu\nu}$ = the flow in the μ direction of momentum in the ν direction

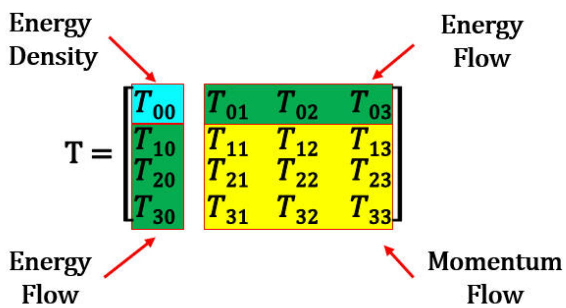
The first component represents classical energy density at a constant time. This was the only component used in Newton's equations.



Similarly, the rest of the top row and left column is the energy flow across each surface.



The rest are momentum flows across surfaces. For example, T_{12} keeps track of the flow in the x direction of momentum in the y direction3. These are caused by pressure and stresses at each surface.



The final step for the gravitational field equations is to determine the constant of proportionality between the Einstein tensor that encapsulates curvature volume and the energy-momentum tensor that encapsulates the total energy density. We use the boundary condition that the equations must produce Newton's equations for spaces with very little curvature. With that, the constant becomes 8 π times Newton's gravitational constant divided by the speed of light raised to the fourth power.

Einstein Field Equations

$$G_{\mu\upsilon} = \frac{8\pi G}{c^4} T_{\mu\upsilon}$$

This looks simple enough, but because they're tensors, it represents 40 equations with 40 unknowns.

Einstein Field Equations

$$G_{\mu\upsilon} = \frac{8\pi G}{c^4} T_{\mu\upsilon}$$

$$\begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} = \frac{8\pi G}{c^4} \begin{bmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{bmatrix}$$

These are the Einstein Field equations for GR. We'll go over what they predict for gravitational phenomena near the Earth, near the Sun and around a Black Hole in our next segment.