How Far Mathematical Foundations – Direct Measurement

*{**Abstract** – A significant amount of mathematics is used in the How Far Away Is It channel video books. Although mathematical equations are identified, they were not the focus. They served to deepen understanding of the physical observations. In this video book, we will begin from first principles and develop the foundation for the math used. But the focus will not be on proofs and notation, but rather on the principles and postulates and key theorems.*

In order to better understand 'direct measurement', we'll develop the real number system from counting numbers. We'll add zero and then negative numbers to get the integer number line. At that point, we introduce the basic mathematical operations of addition, subtraction, multiplication and division. With these we use Peano's Postulates to identify the associative, commutative and distributive properties of whole numbers.

We then extend the integer number line to the rational number line and illustrate the Trichotomy property. We then cover irrational numbers – going back to the ancient Greek philosopher Hippassus – including his proof published by Euclid. Combining sets, we'll construct the dense and continuous real number line, and identify the problems with irrational numbers that persisted until the late 1800s when Richard Dedekind developed his real number line cuts.

And finally, the real number line is then used as the bases for direct measurement where we will identify a key difference between Math and Physics.

Along the way, we'll see: the Bakhshali manuscript where we see the first use of zero; the earliest written reference to negative numbers in the Chinese book "The Nine Chapters on the Mathematical Art"; an algebraic exercise that purports to show that the number one equals the number two; some key issues with the number zero, division and exponents; and a way to multiply by doubling and halving.

This real number line will be the bases for all subsequent mathematical analysis. I trust you'll find it informative and entertaining.}

Introduction

One of the defining characteristics of the "How Far Away Is It" science videos is the limited use of mathematics to shed light on key scientific findings. By limited, I mean the videos focus on the scientific finding much more than the mathematics. But I believe that a deeper understanding of the mathematics could provide for a deeper understanding of the associated scientific conclusions.

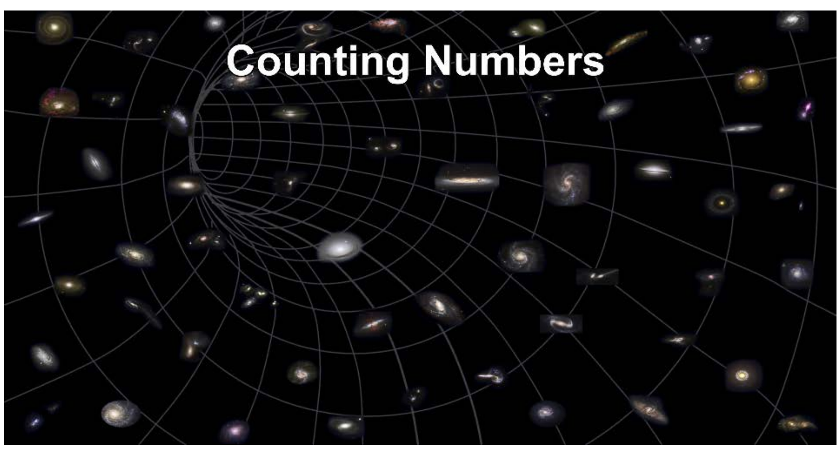
In this video book, we'll cover the math presented in the "How far", "How small", "How fast", and "How old" video books. We won't cover all of mathematics, but rather focus on those aspects that clarify the science.

Counting Numbers

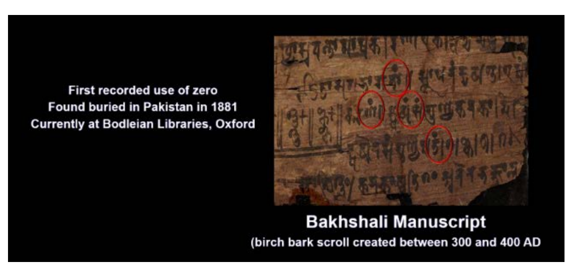
Our first use of mathematics in "How Far Away Is It" was the direct measurement of the distance between where I was standing and a pillar in my backyard. The mathematical foundations for this go to the 'number line' which is based on our concept of numbers. So we'll start with the Number System.

In the beginning across all ancient cultures, there were **natural numbers** or **counting numbers**. Like eggs in a basket, we can have one, or one more than that would be two, or one more than that would be three, etc.

The beauty of mathematics is that it is so versatile. It can not only be used to count eggs, it can count money, or stars in our galaxy, or even galaxies within a range of redshifts to determine whether the universe is flat or not.



The counting numbers did not include the number zero. You cannot count zero. Its origins date back to a famous ancient Indian scroll, called the **Bakhshali manuscript** created 1,600 to 1, 700 years ago. Back then, it was written as a dot used as a place holder for numbers larger than 9. If we add the number zero to our set of counting numbers we get the set of whole numbers.



There are multiple sets of symbols for these numbers. The most familiar are the Roman Numerals, the 10 repeating positional Hindu-Arabic numerals and the 2 repeating positional computer digits. The Hindu-Arabic numerals are by far the most versatile and replaced Roman Numerals when the Roman Empire fell around 300 C.E.

Name	Roman*	Arabic**	Computer***
zero		0	0
one	I	1	1
two	II	2	10
three	III	3	11
four	IV	4	100
five	V	5	101
six	VI	6	110
seven	VII	7	111
eight	VIII	8	1000
nine	IX	9	1001
ten	X	10	1010

The Missing Digit

The Hindu-Arabic number system has some interesting properties. Here's one of them.

Multiply 9 times any number, say 983,264. Pick one of the digits in the product. We'll pick the 7.

$$\begin{array}{r} 983{,}264 \\ \text{x } 9 \\ \hline 8{,}849{,}376 \end{array}$$

Now add up the remaining digits. If the answer has more than one digit, add them. Repeat the process until you get a single digit.

$$\begin{array}{r} 8 \\ 8 \\ 4 \\ 9 \\ 3 \\ +6 \\ \hline 38 \end{array} \qquad \begin{array}{r} 3 \\ +8 \\ \hline 11 \end{array} \qquad \begin{array}{r} 1 \\ +1 \\ \hline 2 \end{array}$$

Subtract that digit from 9 and you get the number you picked out of the original product. This will work no matter which digit you choose to remove from the product.

$$\begin{array}{r} 9 \\ -2 \\ \hline 7 \end{array}$$

It's the base 10 system that makes this work. For example, we can write the number abc,def as the sum of its positional digits.

$$abc{,}def = a(100000) + b(10000) + c(1000) + d(100) + e(10) + f$$

With this view, we can see that the numbers 2, 5 and 10 divide evenly into each of the terms except possibly the last digit - f.

$$abc{,}def = \underbrace{a(100000) + b(10000) + c(1000) + d(100) + e(10)}_{\text{divisible by 2, 5, and 10}} + f$$

This is why, if the last digit is even, the whole number is even; if the last digit is odd, the whole number is odd; if the last digit is divisible by 5, then the whole number is divisible by 5; and if the last digit is 0, the whole number is divisible by 10.

Moving on a bit we can rewrite this number by breaking up the powers of ten subtracting 1 and adding 1 to each of them.

$$a(99999 + 1) + b(9999 + 1) + c(999 + 1) + d(99 + 1) + e(9 + 1) + f$$

Multiplying each digit through its sum

$$a(99999) + a + b(9999) + b + c(999) + c + d(99) + d + e(9) + e + f$$

And rearranging we get this:

$$\underbrace{a(99999) + b(9999) + c(999) + d(99) + e(9)}_{\text{divisible by 9}} + \underbrace{a + b + c + d + e + f}_{\text{sum of the digits}}$$

Every number in the first bracket is divisible by 9. So if the sum of the digits in the second bracket is also divisible by 9, the whole number is divisible by 9. In other words, it's sufficient. Furthermore, if the sum in the second bracket is not divisible by 9, then the whole number is not divisible by 9. In other words it's necessary. By the way, the same thing is true for 3 as well.

So, in our 'missing digit' exercise, we multiplied a number by 9 guarantying that the sum of the product's digits would add up to 9. Now pulling out any digit will reduce the remaining sum by just that amount. So subtracting it from 9 gives you the removed digit.

	8	8	
	8	8	
983,264	4	4	
x 9	9	9	
8,849,376	3	$\cancel{3}$	3 less than 9
	7	7	4
	+6	+6	+2
	45	42	6

This system is easily extended into the decimal number system by dividing by ten for each positon to the right of the decimal point in much the same way we multiplied by ten for position to the left. We write abc.efg as this:

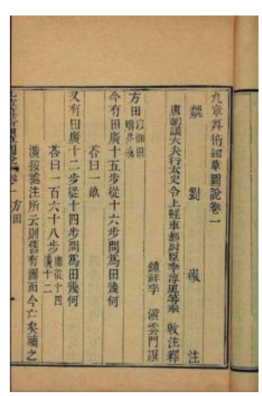
$$a(100) + b(10) + c(1)\ .\ e(1/10) + f(1/100) + g(1/1000)$$

For example 0.75 is $7/10 + 5/100 = 70/100 + 5/100 = 75/100 = 3/4$

Negative Numbers

If we include negative numbers along with positive numbers and zero, we get the full set of **Integers**.

But it took a long time to fully accept the very concept of negative numbers. The ancient Greeks did not have negative numbers. The earliest written reference to negative numbers was found in the Chinese book "The Nine Chapters on the Mathematical Art" written around 100 B.C.E.

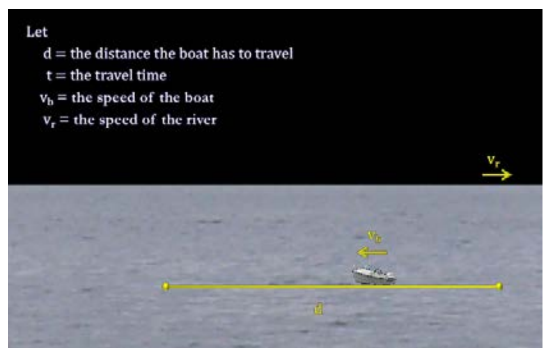


In fact, negative numbers were not fully accepted until the nineteenth century. After all, you can't have less than zero eggs in the basket. [In the 17th century Blaise Pascal, one of the great mathematicians of his time found the statement 0 – 4 to be utterly absurd. And Descartes found negative solutions to equations to be "false" solutions.

Another mathematician argued against negative numbers by assuming that numbers represent quantities. That would have the number 1 representing a larger quantity than the number -1. So to say that the ratio of 1 over -1 (a large quantity over a smaller quantity) is equal to the ratio of -1 over 1 (a smaller quantity over a larger quantity) is absurd. And even Leibniz, who created Calculus at the same time as Newton in the 18th century regarded this objection to negative numbers as valid. But he used them none the less.]

Here's an illustration that highlights the problem mathematicians had with negative numbers. We used this in the "How Fast Is It" video book explaining the theory behind the Michelson-Morley experiment.

We have a boat in a river traveling upstream with a motor that can drive it at a steady speed in still water. The river is flowing in the opposite direction. The boat's home base is a known distance away. The question is - How long will it take the boat to get home?



The solution is pretty straight forward. The time it takes is just the distance it has to travel divided by the speed it is traveling. And that speed would be its velocity minus the velocity of the river.

$$t = d/(v_b - v_r)$$

If the distance is 30 km, and the boat is running at 20 km/hr, and the current working against it is 5 km/hr we see that the trip home will take 2 hours.

$$t = 30 \text{ km}/(20\text{km/hr} – 5 \text{ km/hr}) = 2 \text{ hr}$$

But what if the current is greater than the speed of the boat say 25 km/hr. Then the equation gives us negative time. Is time going backwards? Absurd.

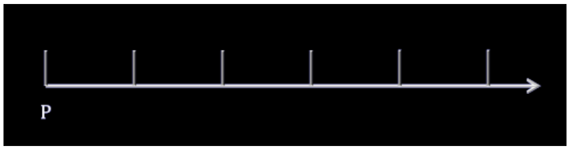
$$t = 30 \text{ km}/(20\text{km/hr} – 25 \text{ km/hr}) = -6 \text{ hr}$$

But if we apply the math to the situation that we used to develop the equation, we see that a negative time simply means that the poor boat will never make it home. The river will simply continue to carry it downstream.

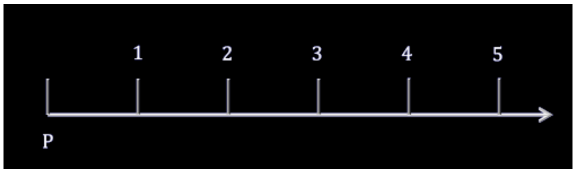
The Number Line

With the full set of integers, we can now construct the number line in order to use numbers to measure distances. We need to associate each number to a point on a line.

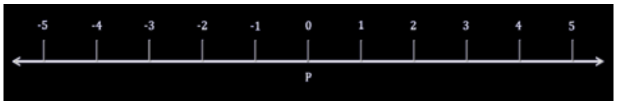
To construct a correspondence between positive integers and points on a line, we begin by marking off equal segments to the right of the origin P of a given line.



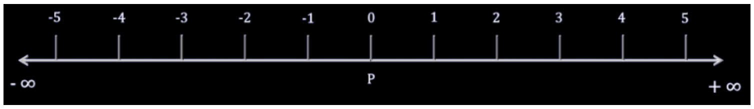
We associate the number 1 with the right endpoint of the first segment. 2 with the right endpoint of the second segment, etc. This method associates each positive integer with a point on the line. The number associated with a point on the line is called its coordinate.



We then associate the number zero with the origin P, and then extend the line to the left in the same sized segments we count off to the right. These points correspond to the negative numbers where the leftmost point in the first segment represents -1, the leftmost point in the second segment represents -2, and so forth. This is the basic number line for the set of all integers.



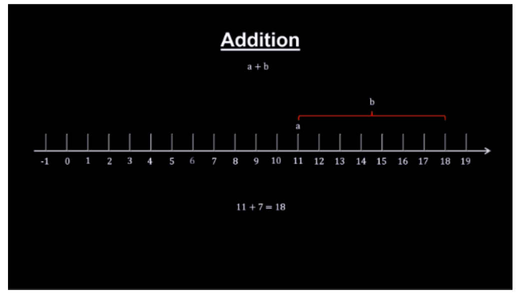
In order to indicate that these numbers are carried out to the right and the left without limit, we introduce the symbols $+\infty$ and $-\infty$. But we need to keep in mind that these are not numbers.



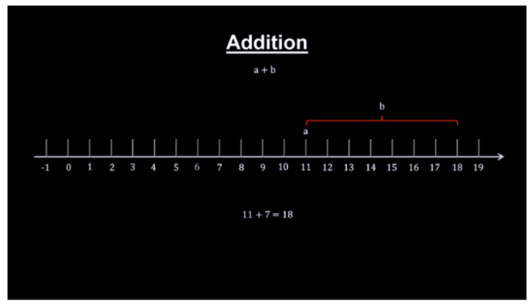
Basic Arithmetic Operations

With the set of integers in hand, we can define the four basic arithmetic operations of addition, subtraction, multiplication and division.

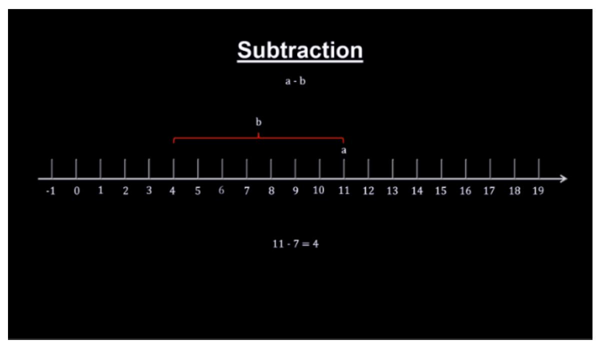
We can define addition as an exercise in counting. a + b means start with the number specified by the first term ‘a’ and count the number specified by the second term ‘b’. In this example, we start with ‘a’ at 11 and count 7 more for ‘b’.



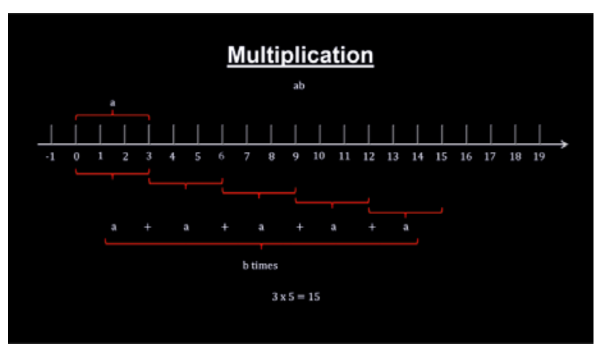
Zero is the additive identity in that adding zero to any number gives you the same number you started with.



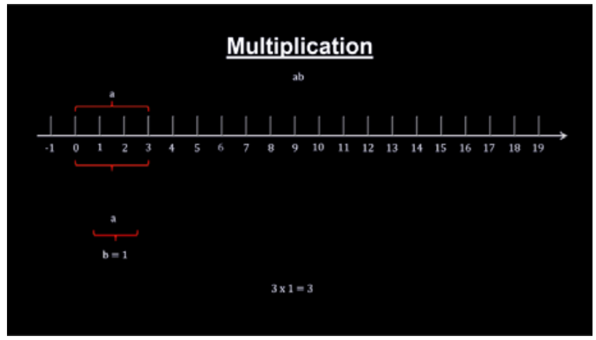
Subtraction would then be start with the first term and count backwards the number specified by the second term. Here we are counting 7 numbers back from 11. Thus addition and subtraction are tied directly to counting, and counting has been shown to accurately represent anything that you can count.



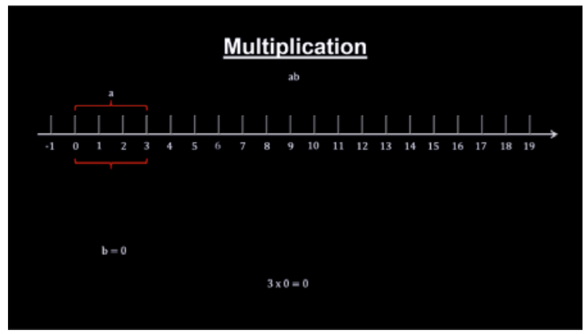
We can also define multiplication in terms of addition which in turn is based on counting. ‘a’ times ‘b’ says add the number ‘a’ to itself ‘b’ times. Here we are starting with 3 and adding it to itself 5 times. [In the appendix, you’ll find an example of multiplication that just uses doubling and halving.]



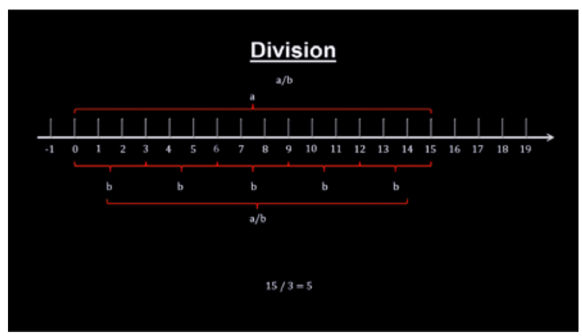
Doing it 1 time leaves it unchanged. Thus, the number 1 is the multiplicative identity like zero was the additive identify.



But what does it mean to add a number to itself zero times. To deal with that we define a number being added to itself zero times, to be the number zero.



We define division as the inverse of multiplication. For the division of one number (say ‘a’) by another number (say ‘b’), we are asking how many times can we subtract b from a. Or more generally, what number when multiplied by ‘b’ gives us ‘a’. For example, in the above multiplication we could ask how many times can we subtract 3 from 15, and the answer is 5.



In the late 1800s, the mathematician Giuseppe Peano proposed a set of axioms or postulates that can be used to develop number properties. In simple terms, starting with the counting numbers, they are:

1. **1 is a number**
2. **Every number n has one and only one successor number n + 1.**
3. **No two different numbers have the same successor number**
    - **If n + 1 = m +1 then n = m.**

From these postulates and our basic operator definitions, a number of properties exist that we can use to manipulate numbers and solve equations. Here are the properties for the natural or counting numbers. Although we tend to take them for granted, mathematicians have to have proven each and every one.

Natural Number Properties

Let $\mathbb{N}$ be the set of Natural Numbers	$\mathbb{N} = \{x \mid x \text{ is a natural number}\}$
• $\mathbb{N}$ is closed with respect to addition	If $a \in \mathbb{N}$ and $b \in \mathbb{N}$, then $a + b \in \mathbb{N}$
• $\mathbb{N}$ is not closed with respect to subtraction	If $a \in \mathbb{N}$ and $b \in \mathbb{N}$, then $a - b$ may not be in $\mathbb{N}$
• Zero is the additive identity	$a + 0 = a$
• Addition is Associative	$(a + b) + c = a + (b + c)$
• Addition is Commutative	$a + b = b + a$
• $\mathbb{N}$ is closed with respect to Multiplication	If $a \in \mathbb{N}$ and $b \in \mathbb{N}$, then $ab \in \mathbb{N}$
• $\mathbb{N}$ is not closed with respect to division	If $a \in \mathbb{N}$ and $b \in \mathbb{N}$, then a/b may not be in $\mathbb{N}$
• One is the multiplicative identity	$a \times 1 = a$
• Multiplication is Associative	$(ab)c = a(bc)$
• Multiplication is Commutative	$ab = ba$
• Multiplication is distributive over addition	$a(b + c) = ab + ac$

Natural numbers are closed for addition and multiplication. By closed we mean that these operations on numbers in the set produce numbers that are also in the set. When we include zero and negative numbers we get the integer number line. The set of integers adds closure for subtraction.

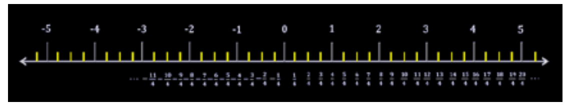
Let $\mathbb{Z}$ be the set of Integers	$\mathbb{Z} = \{x \mid x \text{ is an integer}\}$
• $\mathbb{Z}$ is closed with respect to subtraction	If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$, then $a - b \in \mathbb{Z}$

You'll find a proof that $(a + b) = (b + a)$ for natural numbers in the appendix.

The Rational Number Line

The set of integers is not closed with respect to division. 1 divided by 2 is not in the set of integers. To include these and make the set closed with respect to division, we need to add all the rational numbers - the numbers that can be expressed as a ratio of integers where the denominator is not zero.

To map these numbers to the number line, we simply divide each segment into the number of subsegments indicated by the denominator. For example, here's 9 divided by 4.



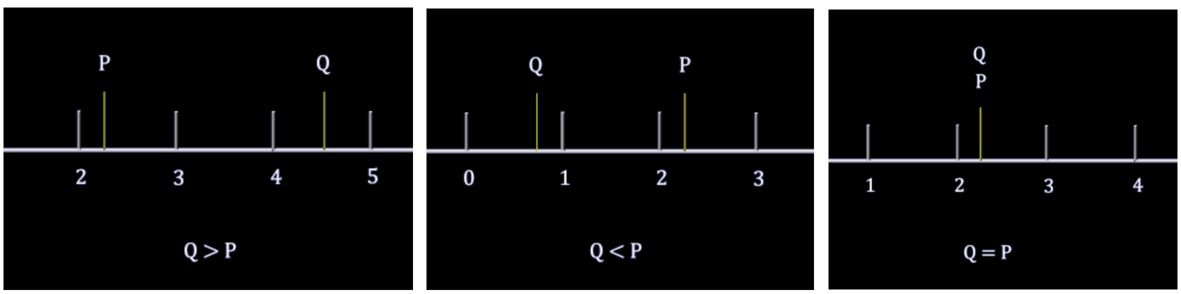
The line that contains all integers and rational numbers and zero is known as the Rational Number Line. It has all the properties of the integer number line and is closed for all the basic operations.

Let ℚ be the set of rational numbers **ℚ = {x| x is a rational number}**

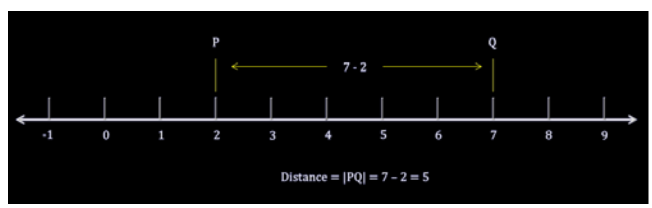
- **ℚ is closed with respect to division** **If a ∈ ℚ and b ∈ ℚ, then a/b ∈ ℚ**

It not only contains all the rational numbers, it provides the ordering. One number is greater than another if its coordinate on the line is to the right, it is less than another if its coordinate is to the left, and it is equal to the other if its coordinate is the same as the other.

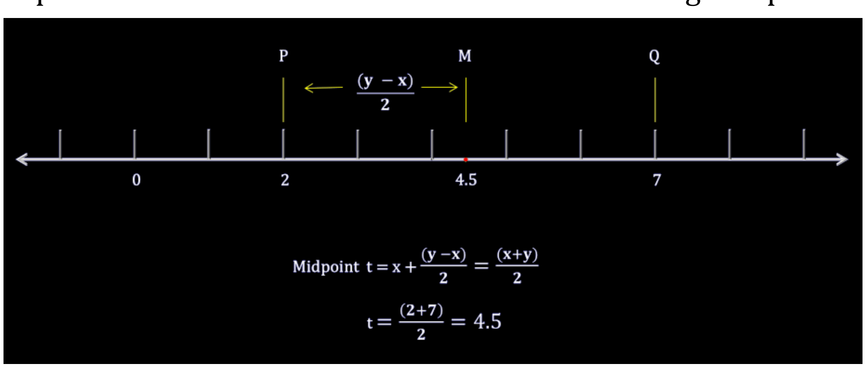
Trichotomy property



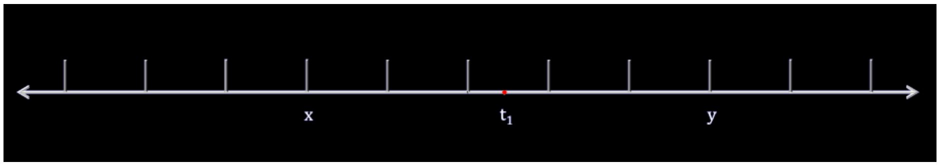
With the rational number line in hand, we can now define **distance** on the line. If P and Q are rational points on the line with coordinates x and y respectively such that $x \leq y$, then the distance between P and Q is $(y - x)$. [For example, if the coordinate for P is 2 and the coordinate for Q is 7, then the distance from P to Q is 5.]



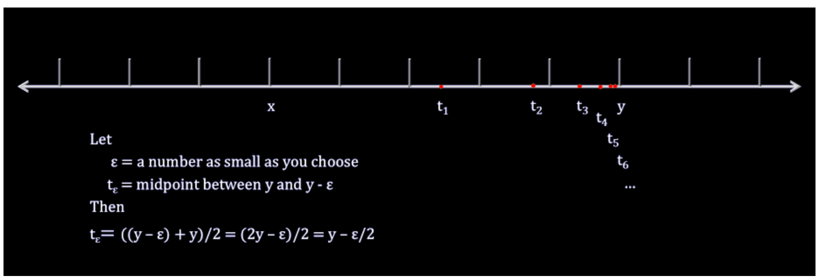
It is straight forward to find the midpoint between any two rational points P and Q. We'll call it M with a coordinate equal to t. The distance to t would be the distance to P plus half the distance between P and Q. That would be $(y - x)/2$. We see that the coordinate of the midpoint of two points on the number line is half the sum of the given points.



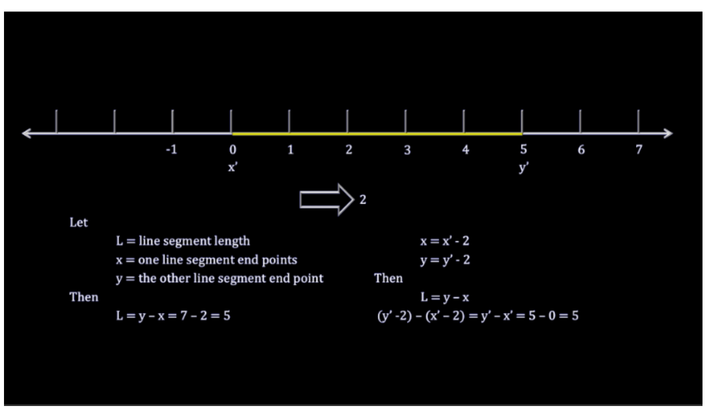
Consequently, there is always a rational point between any two rational points on the line. Because we can continue to divide by 2 without limit, it follows that there are infinitely many rational numbers between any two given rational numbers no matter how close together they are. We say the rational number line is **dense**. [This also means that there is no such thing as adjacent points – two points without any other points between them.]



An important consequence of the set of rational numbers being dense is that the length of any segment can be approximated to any degree of accuracy by a rational number.

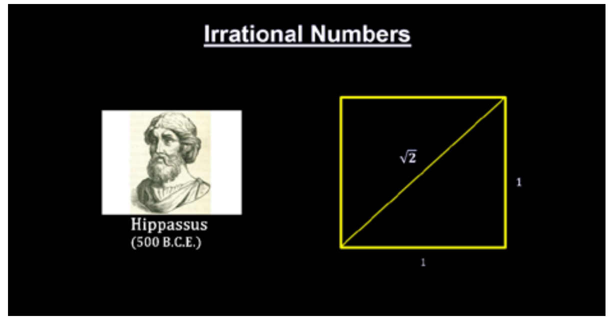


And there is one more important point to make about this line. If we shift the origin, every coordinate on the line will change. For example, if we shift it to the right 2 units, the coordinates on the new line will be different than the originals by 2 units. But there is one thing that does not change when we change the coordinates, and that is the length of any line. It is said to be invariant with respect to cardinalate transformations. You can see how the shift in the values of the coordinates cancel out in the length calculation.



But before we leave the number line, there are two more considerations we need to examine: 1) there are missing points on the rational number line and 2) the number zero has some unique important characteristics.

Irrational Numbers



Hippassus born around 500 B.C.E. was a Greek philosopher of the Pythagorean school of thought. He is widely regarded as the first person to recognize that a square's diagonal cannot be expressed as the ratio of two integers. At this time in Greek society, numbers were intimately connected to their religion, so Hippassus' finding was considered heresy.

Two hundred years later, Euclid published the proof. Here's how it goes. First, assume that there is such a rational number, and then show a resulting contradiction that negates the assumption.

Suppose p/q is a rational number expressed in its lowest terms (meaning they have no common factors except for the number 1 such that

$$\frac{p}{q} = \sqrt{2}$$

Squaring both sides of the equation and multiplying both sides by q^2 we get

$$p^2 = 2q^2$$

This shows that p^2 is an even number and therefore, p must be an even number because an odd number times itself would be an odd number.

Since p is even there exists a number t such that

$$p = 2t$$

If we substitute this in for p, we get

$$(2t)^2 = 2q^2$$

$$4t^2 = 2q^2$$

If we divide both sides by 2 we get this

$$2t^2 = q^2$$

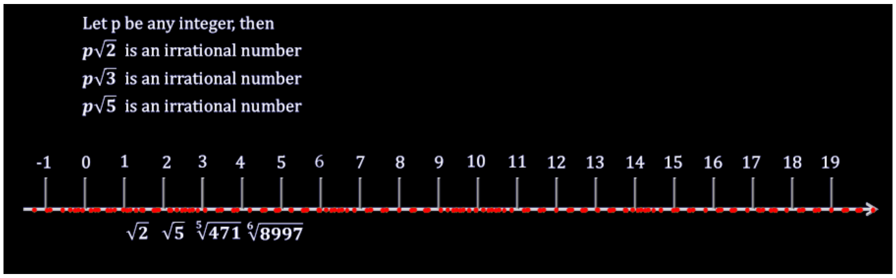
Showing that q is also an even number. In other words both p and q have 2 as a factor.

But our stipulation was that they had no factors in common except for the number 1. We have a contradiction. And it shows that the statement "The square root of 2 can be expressed as a rational number." is false. Therefore, it cannot be expressed as a rational number.

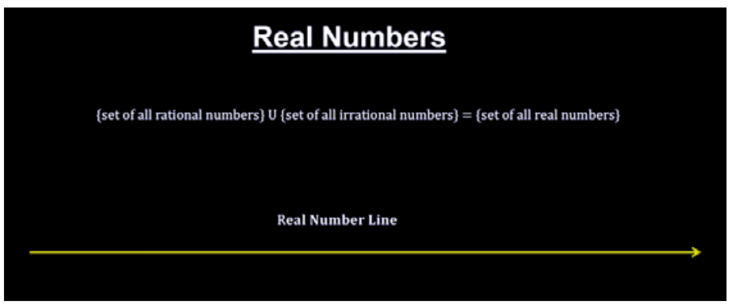
In fact, the nth root of any number that isn't a perfect nth root is irrational.

$$\sqrt{5}\ ,\sqrt[5]{471}\ ,\ \sqrt[6]{8997}\ ,\ \ldots$$

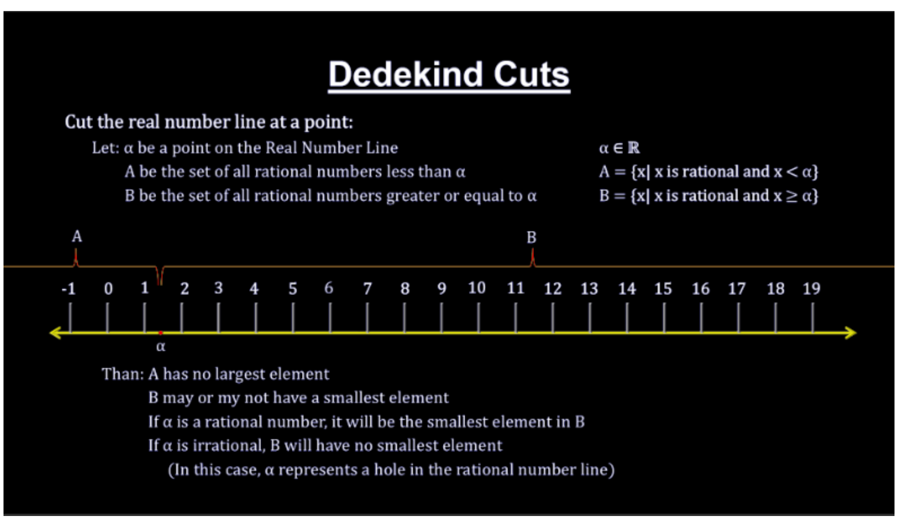
Add to that the fact that any irrational times a rational will be irrational and you can see that the set of irrational numbers is infinite. That puts a lot of holes in the Rational Number Line. The rational number line might be dense, but it is not continuous.



The union of the set of rational numbers and irrational numbers creates the set of Real Numbers.



But to prove the basic number properties for a number line that includes irrational numbers turned out to be quite the problem. In 1872, Richard Dedekind defined cuts in the rational number line that exposed the holes created by irrational numbers. He then proves that the set of these cuts is equivalent to the set of real numbers.



This extended the rational number line into the real number line in a manner that preserved all the properties of the rational number line. In addition, it is not only dense, it is continuous. It has no holes. The Real Number Line is the foundation from which all the rest of our math will flow.

One Equals Two

Here's a basic algebraic exercise that illustrates the issue covered in the next segment.

Let $a = 1$, and $b = 1$. Then $\quad a = b$

We can multiply both sides of this equation by a and get $\quad a^2 = ab$

We can subtract b^2 from both sides and get $\quad a^2 - b^2 = ab - b^2$

We can factor $a^2 - b^2$ into $(a + b)(a - b)$ on the left hand side of the equation and $ab - b^2$ into $b(a - b)$ on the right side of the equation giving us $\quad (a + b)(a - b) = b(a - b)$

We can multiply both sides by $1/(a - b)$ and get $\quad (a + b)(a - b)/(a - b) = b(a - b)/(a - b)$

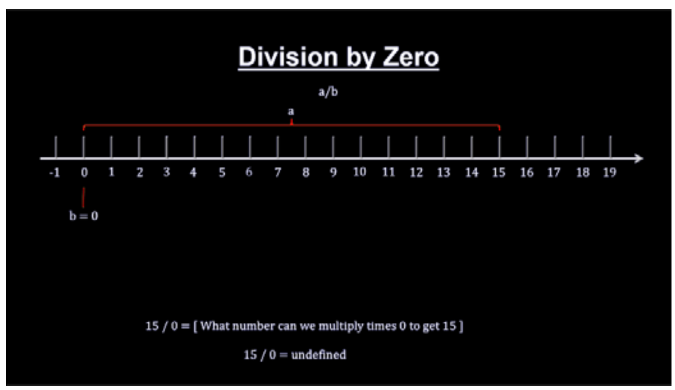
The a-b terms cancel out. So the equation simplifies to $\quad (a + b) = b$

Substituting 1s in for a and b we get $\quad 1 + 1 = 1$ or $2 = 1$

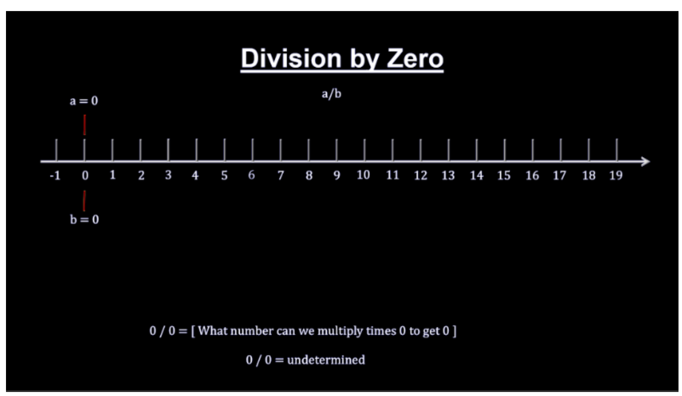
This happened because we divided both sides of the equation by $(a - b)$ which equals zero. In this way dividing by zero is like a box of chocolates – you never know what you're going to get. A closer look at the number zero will explain why.

The Trouble with Zero

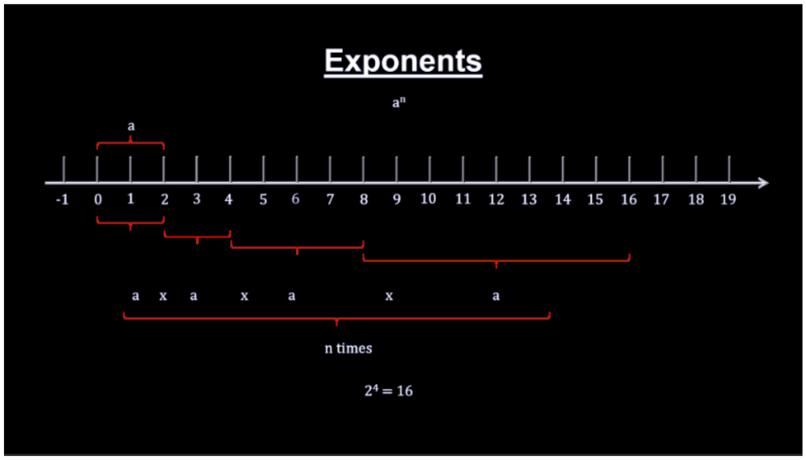
It's important to understand that division has a problem when it comes to the number 0. Suppose 'a' is a number not equal to 0. Then, for a/0 we're asking what number, when multiplied by 0 will give us 'a'. But no matter what number you multiply by 0, you will always get 0, never 'a'. So a/0 has no meaning. We say it is undefined.



Now, if $a = 0$, we're asking what number when multiplied by 0 would give us 0. The answer is any number at all because any number multiplied by 0 would give you 0. This makes 0/0 completely undetermined. It can be any number you can think of. This is what gave us the 1 equals 2 result.



The problem with zero also shows up with exponents as well. For any number a and any positive integer n, we define a (the base) raised to the n^{th} power (the exponent) to be the base multiplied by itself the number of times specified by the exponent.



We see that, when we multiply two numbers with the same base, we can add the exponents:

$$a^n \text{ x } a^m = a^{(n+m)} \qquad\qquad 2^2 \text{ x } 2^3 = 2\text{x}2 \text{ x } 2\text{x}2\text{x}2 = 2^5$$

With that in mind, we define a negative exponent to mean one dived by the base raised to the positive value of the exponent:

$$a^{-n} = 1/a^n \qquad\qquad 3^{-2} = 1/3^2$$

This extends the exponent arithmetic to include all integers. We see that

$$a^n \text{ x } a^{-m} = a^{(n-m)} \qquad\qquad 4^5 \text{ x } 4^{-2} = 4\text{x}4\text{x}4\text{x}4\text{x}4/4\text{x}4 = 4\text{x}4\text{x}4 = 4^3$$

It follows that a raised to the nth power times a raised to the –nth power will equal the number one:

$$a^n \text{ x } a^{-n} = a^n/a^n = 1$$

It also follows that a raised to the nth power times a raised to the –nth power will equal a raised to the zero power: So, in order for the arithmetic to hold, we must define a number multiplied by itself zero times to equal the number one – the multiplicative identity. (This is much like adding a number to itself zero times is equal to zero – the additive identity.)

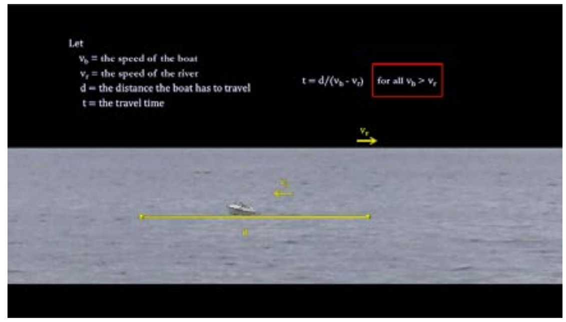
$$a^n \text{ x } a^{-n} = a^{(n-n)} = a^0 = 1$$

But we also know that if the base is zero, raising it to any power will always give you zero:

$$0^n = \underbrace{0 \text{ x } 0 \text{ x } 0 \text{ x } 0 \text{ x } 0 \ldots}_{\text{n times}} = 0$$

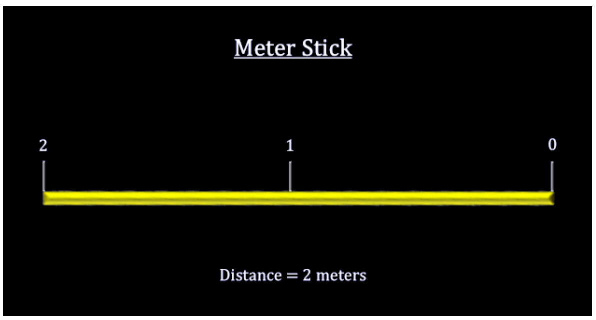
So what if both the base and the exponent are zero? Does $0^0 = 1$ or $0^0 = 0$. It is said to be indeterminate.

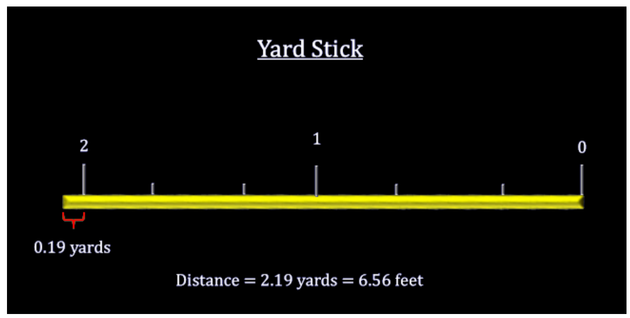
So, when we apply math to a physical situation, we must always take care to never wind up dividing by zero. We must always stipulate the ranges were an equation is operative and where it is not.



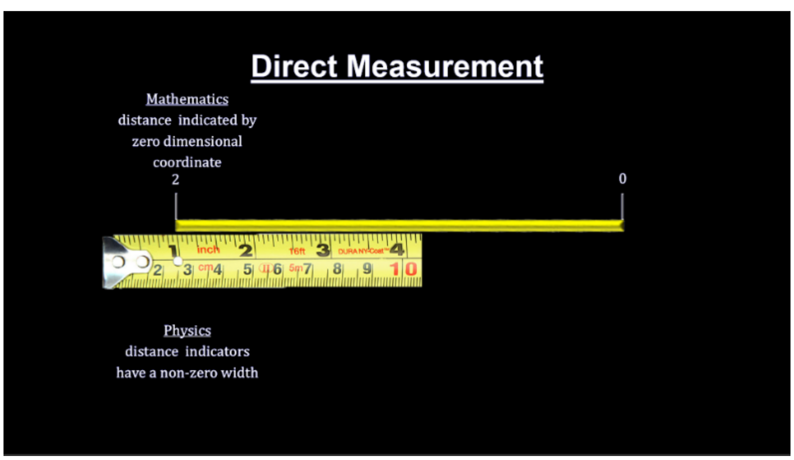
Direct Measurement

We can now take a look at the measurement we started with. If we mark the equal segments on the measuring tape to be meters, we get 2 meters. If we mark the equal segments to be feet, we get 6 and 56/100 feet.





Here we need to highlight a key difference between the pure mathematics and the physics of measuring distances. Math uses exact coordinates to give us the exact distances. But physical measurements always involve a level of inaccuracy. For completeness purposes, you will often see a distance accompanied by an estimated error magnitude. In this case, we would say that the distance to the pillar is 2 meters plus or minus 1 mm (or 2.19 yards plus or minus some fraction of an inch) depending on the accuracy of the tape measure.



Appendix

Multiply by Doubling and Halving

We defined multiplication in terms of addition. This leads to some interesting results as well. Here's a way to do multiplication using doubling, halving and adding. To illustrate, let's multiply 127 by 46 the old way. First we multiply by 6 and then by 40. We add the two together for the result.

$$\begin{array}{r} 127 \\ \underline{\times\,46} \\ 762 \\ \underline{+\,5080} \\ 5842 \end{array}$$

In this method we need two columns. Now pick one of the two numbers to be multiplied – we'll use the 46. Put it at the top of the first column. Divide it in half. If it were an odd number, we'd subtract 1 to get an even number and then divide it in half. Continue this process until you get to the number 1. Now place the other number (127 in our example) at the top of the second column. Double it and double it again and again until you reach the row with the number 1 in the first column. Now scratch out each row that has an even number in the first column and add the remaining numbers in the second column. This is the product.

~~46~~	~~127~~
23	254
11	508
5	1016
~~2~~	~~2032~~
1	4064
	5842

Prove (a + b) = (b + a)

It is quite common in mathematics to use inductive reasoning for proofs of the sort where you're trying to prove something is true for all numbers in an infinite set. Proving the communitive property for addition (a + b) = (b + a) for all counting numbers is one of them.

All we have to go on are our postulates and one proven theorem. The postulates are

- 1 is a number
- Every number n has one and only one successor number n + 1:
- No two different numbers have the same successor number
    - If n + 1 = m +1 then n = m.

And the associative property of addition theorem:

- (a + b) + c = a + (b + c)

The communitive property proof has two parts. In the first part, we show that

$(a + b) = (b + a)$ for any value of 'a' when 'b' equals 1. Our inductive assumption is that for some value of 'a' the relationship is true. We'll show that that implies that the relationship is true for $(a + 1)$.

$(a + 1) + 1$

$= (1 + a) + 1$ by our inductive assumption

$= 1 + (a + 1)$ by the associative property of addition

Now we'll show that $(a + 1) = (1 + a)$ when $a = 1$.

$(a + 1)$

$= (1 + 1)$ because $a = 1$

$= (1 + a)$ because $a = 1$

So it is true for $a = 1$ and by the first result, we know that it is true for $a = 1 + 1 =2$ and then for $a = 2 + 1 = 3$, etc. for all counting numbers.

Now in part two, our inductive assumption is that $(a + b) = (b + a)$ for some value of b. We'll show that this implies that $(a + (b + 1)) = ((b + 1) + a)$

$a + (b + 1)$

$= (a + b) + 1$ by the associative property

$= (b + a) + 1$ by the inductive assumption

$= b + (a + 1)$ by the associative property

$= b + (1 + a)$ by the results of part 1: $a + 1 = 1 + a$

$= (b + 1) + a$ by the associative property

We're done.

Credits

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MIT Calculus online textbook

https://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/instructor-s-manual/

Greek letters:
- α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ σ τ υ φ χ ψ ω
- Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ Υ Φ Χ Ψ Ω

⇒ → ± ⊙ ∞ ↛ ∃ ∄ ∈ ∉ ∬ ∫ ≅ ≥ ≤ ≈ ≠ ≡ √ ∛

ℝ ℤ ℚ ℕ

R = real numbers, Z = integers, N=natural numbers, Q = rational numbers, P = irrational numbers. • ⊂= proper subset (not the whole thing) ⊆=subset • ∃ = there exists • ∀ = for every • ∈= element of • S = union (or) • T = intersection (and) • s.t.= such that • =⇒ implies • ⇐⇒ if and only if • P = sum • \ = set minus • ∴= therefore